

**BOLTZMANN LIMIT FOR A HOMOGENOUS FERMI GAS
WITH DYNAMICAL HARTREE-FOCK INTERACTIONS IN A
RANDOM MEDIUM**

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ABSTRACT. We study the dynamics of the thermal momentum distribution function for an interacting, homogenous Fermi gas on \mathbb{Z}^3 in the presence of an external weak static random potential, where the pair interactions between the fermions are modeled in dynamical Hartree-Fock theory. We determine the Boltzmann limits associated to different scaling regimes defined by the size of the random potential, and the strength of the fermion interactions.

1. INTRODUCTION

We study the dynamics of an interacting, homogenous Fermi gas on \mathbb{Z}^3 in a static, weakly disordered random medium, where the pair interactions between the fermions are modeled in dynamical Hartree-Fock theory. An observable of considerable importance is the momentum distribution function at positive temperature, and we are interested in its dynamics for time scales that are associated to kinetic scaling limits of Boltzmann type. A main motivation is to understand the trend to equilibrium in such systems, and to control the interplay between the influence of the static randomness, and nonlinear self-interactions of the particles. We derive Boltzmann limits for the thermal momentum distribution function, depending on different scaling ratios between the random potential, and the strength of the pair interactions between the fermions.

The model in discussion describes a gas of fermions in a finite box $\Lambda_L := [-\frac{L}{2}, \frac{L}{2}]^3 \cap \mathbb{Z}^3$ of side length $L \gg 1$ with periodic boundary conditions, and associated dual lattice $\Lambda_L^* := \Lambda_L/L \subset \mathbb{T}^3$; we will eventually take the thermodynamic limit $L \rightarrow \infty$. On the fermionic Fock space of scalar electrons, $\mathfrak{F} = \mathbb{C} \oplus \bigoplus_{n \geq 1} \bigwedge_1^n \ell^2(\Lambda_L)$, we introduce creation- and annihilation operators a_x^+, a_y , for $x, y \in \Lambda_L$, satisfying the canonical anticommutation relations $\{a_x^+, a_y\} := a_x^+ a_y + a_y a_x^+ = L^3 \delta_{p,q}$ where $\delta_{p,q}$ is the Kronecker delta, and $\{a_x^\sharp, a_y^\sharp\} = 0$ for $a^\sharp = a$ or a^+ . We denote the Fock vacuum by $\Omega = (1, 0, 0, \dots) \in \mathfrak{F}$; it is annihilated by all annihilation operators, $a_x \Omega = 0$ for all $x \in \Lambda_L$.

Letting \mathfrak{A} denote the C^* -algebra of bounded operators on \mathfrak{F} , we consider a time-dependent state ρ_t on \mathfrak{A} determined by

$$i\partial_t \rho_t(A) = \rho_t([H(t), A]) \quad (1.1)$$

for $A \in \mathfrak{A}$, with a translation invariant initial condition ρ_0 . We assume that ρ_0 is number conserving $\rho_0([A, N]) = 0$ for all $A \in \mathfrak{A}$, where $N := \sum_{x \in \Lambda_L} a_x^+ a_x$ denotes

the particle number operator. We study the dynamics of ρ_t determined by the time-dependent Hamiltonian

$$H(t) = H_0 + \eta V_\omega + \lambda W(t), \quad (1.2)$$

where $H_0 := \int dp E(p) a_p^+ a_p$ is the second quantization of the centered nearest neighbor Laplacian, with $E(p) = 2 \sum_{j=1}^3 \cos(2\pi p_j)$ denoting its symbol. The interaction of the fermion gas with the static random background potential is described by the operator $V_\omega := \sum_{x \in \Lambda_L} \omega_x a_x^+ a_x$ where $\{\omega_x\}_{x \in \Lambda_L}$ is a field of i.i.d. centered, normalized, Gaussian random variables. The small parameter $0 < \eta \ll 1$ determines the strength of the disorder.

The operator $W(t)$ accounts for fermion pair interactions in dynamical Hartree-Fock theory,

$$W(t) := \sum_{x,y \in \Lambda_L} v(x-y) \left[\mathbb{E}[\rho_t(a_x^+ a_x)] a_y^+ a_y - \mathbb{E}[\rho_t(a_y^+ a_y)] a_x^+ a_x \right]. \quad (1.3)$$

Here, v denotes a pair potential, where $\|\widehat{v}\|_{H^{3/2+\sigma}(\mathbb{T}^3)} < C$ is assumed for $\sigma > 0$ arbitrary but fixed. Notably, the unknown quantity $\mathbb{E}[\rho_t(a_x^+ a_x)]$ itself appears in $W(t)$.

Our main interest is to study the dynamics of $\mathbb{E}[\rho_t(a_y^+ a_x)]$, the average of the pair correlation function, which is determined by the self-consistent nonlinear evolution equation

$$i\partial_t \mathbb{E}[\rho_t(a_x^+ a_y)] = \mathbb{E}[\rho_t([H(t), a_x^+ a_y])] \quad (1.4)$$

with initial condition $\rho_0(a_x^+ a_y)$. In particular, we derive Boltzmann equations in kinetic scaling limits of the above model, for scaling regimes defined by different ratios between η and λ .

The relevant scaling relations in the system can be understood with the help of the following heuristics. The assumption of vanishing mean implies that the average effect of the random potential on the dynamics of μ_t in a time interval $[0, t]$ is proportional to its variance, of size $O(\eta^2 t)$. This suggests that the strength of the pair interactions between fermions, and the interactions of each fermion with the random potential, are comparable if $\lambda = O(\eta^2)$. Accordingly, we distinguish the following scaling regimes, for which we derive the associated Boltzmann limits:

- The scaling regime $\lambda = O(\eta^2)$, where the interactions between pairs of fermions, and of each fermion with the random potential are comparable. For any $T > 0$, for test functions f, g , and $T/t = \eta^2$, we prove

$$\lim_{\eta \rightarrow 0} \lim_{L \rightarrow \infty} \mathbb{E}[\rho_{T/\eta^2}(a^+(f)a(g))] = \int dp \overline{f(p)} g(p) F_T(p), \quad (1.5)$$

where $a^+(f) = \sum_x f(x) a_x^+ = (a(f))^*$ (adjoint), and $F_T(p)$ satisfies the linear Boltzmann equation

$$\partial_T F_T(p) = 2\pi \int du \delta(E(u) - E(p)) (F_T(u) - F_T(p)) \quad (1.6)$$

with initial condition $F_0 = \mu_0 \in H^{\frac{3}{2}+\sigma}(\mathbb{T}^3)$ for some $\sigma > 0$. The fact that the resulting Boltzmann equation is linear follows from complicated phase cancellations due to translation invariance. While the microscopic

dynamics is nonlinear, we prove that its kinetic scaling limit, as $\eta \rightarrow 0$, is described by a *linear* Boltzmann equation. The scattering kernel accounts for elastic collisions preserving the kinetic energy. This result remains valid in the regime $\lambda = o(\eta^2)$. In the case $\lambda = 0$, it reduces to the one proven in [10].

- In the regime $\eta^2 = o(\lambda)$ and $(T, X) = (\lambda t, \lambda x)$, we prove that the kinetic scaling limit $\lambda \rightarrow 0$ is stationary.
- In the regime where $\lambda > 0$ is independent of η , and for the scaling $(T, X) = (\eta^2 t, \eta^2 x)$, we characterize the stationary states; those are given by solutions of an implicit equation of the form (3.4) with zero on the l.h.s., but where the delta distribution enforces conservation of a *renormalized* energy per particle. Accordingly, the stationary states are supported on level surfaces of a *renormalized* kinetic energy function, determined by a nonlinear fixed point equation. A derivation of non-stationary solutions in this kinetic scaling limit is an interesting open problem.

Our work significantly extends [10] which addresses the Boltzmann limit for a homogenous free Fermi gas in a random medium. The proofs given in [10] employ techniques developed [8, 9, 17, 12] developed for the derivation of Boltzmann equations from the quantum dynamics of a single electron in a weak random potential; see also [22, 26]. In the landmark works [13, 14, 15], this analysis has been extended to diffusive time scales. We also refer to [1, 6, 7, 11, 20, 25] for related works.

For the proof of our results, we represent the average momentum density $\mathbb{E}[\mu_t]$ in integral form, as an expansion organized in terms of Feynman diagrams. Our overall strategy parallels the approach in [17, 8, 10, 12, 9, 22, 26], but the techniques developed in those works (for linear models) do not carry over directly because of the nonlinear self-interaction of the fermion field. In particular, resolvent methods which underlie the analysis those works are not available here. Instead, our approach strongly uses stationary phase techniques, in order to control the combination of Feynman graph expansion techniques with such nonlinearities.

For the related problem of the derivation of dynamical Hartree-Fock equations from a fermion gas, see for instance [4]. We note that the derivation of macroscopic transport equations from the quantum dynamics of Fermi gases without any simplifying assumptions on the interparticle interactions (and without random potential) is a prominent and very challenging open problem in this research field; see for instance [5, 16, 18, 23, 27]. For some very interesting recent progress relevant related to this issue, see [24].

2. DEFINITION OF THE MODEL

We consider a gas of fermions in a finite box $\Lambda_L := [-\frac{L}{2}, \frac{L}{2}]^3 \cap \mathbb{Z}^3$ of side length $L \gg 1$, with periodic boundary conditions. We denote the associated dual lattice by $\Lambda_L^* := \Lambda_L/L \subset \mathbb{T}^3$. We assume that L is much larger than any other significant length scale of the system, which will depend upon the case under consideration. We will eventually take the thermodynamic limit $L \rightarrow \infty$.

We denote the Fourier transform by

$$\widehat{f}(p) := \sum_{x \in \Lambda_L} e^{-2\pi i p \cdot x} f(x), \quad (2.1)$$

where $p \in \Lambda_L^*$, and the inverse transform by

$$g^\vee(x) = \int dp e^{2\pi i p \cdot x} g(p). \quad (2.2)$$

For brevity, we are using the notation

$$\int dp f(p) \equiv \frac{1}{L^3} \sum_{p \in \Lambda_L^*} f(p), \quad (2.3)$$

which recovers its usual meaning in the thermodynamic limit $L \rightarrow \infty$.

We will use the notation

$$\delta(k) := L^3 \delta_k, \quad (2.4)$$

where

$$\delta_k = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases} \quad (2.5)$$

denotes the Kronecker delta on the momentum lattice $\Lambda_L^* \pmod{\mathbb{T}^3}$.

We denote the fermionic Fock space of scalar electrons by

$$\mathfrak{F} = \bigoplus_{n \geq 0} \mathfrak{F}_n, \quad (2.6)$$

where

$$\mathfrak{F}_0 = \mathbb{C}, \quad \mathfrak{F}_n = \bigwedge_1^n \ell^2(\Lambda_L), \quad n \geq 1. \quad (2.7)$$

We introduce creation- and annihilation operators a_p^+, a_q , for $p, q \in \Lambda_L^*$, satisfying the canonical anticommutation relations

$$a_p^+ a_q + a_q a_p^+ = \delta(p - q) := \begin{cases} L^3 & \text{if } p = q \\ 0 & \text{otherwise.} \end{cases} \quad (2.8)$$

There is a unique unit ray $\Omega = (1, 0, 0, \dots) \in \mathfrak{F}$, referred to as the Fock vacuum, which is annihilated by all annihilation operators, $a_p \Omega = 0$ for all $p \in \Lambda_L^*$.

Let \mathfrak{A} denote the C^* -algebra of bounded operators on \mathfrak{F} . We consider a time-dependent state ρ_t on \mathfrak{A} determined by

$$i\partial_t \rho_t(A) = \rho_t([H(t), A]) \quad (2.9)$$

for $A \in \mathfrak{A}$, with a translation invariant initial condition ρ_0 . We assume that ρ_0 is number conserving; that is,

$$\rho_0([A, N]) = 0 \quad (2.10)$$

for all $A \in \mathfrak{A}$, where

$$N := \sum_{x \in \Lambda_L} a_x^+ a_x \quad (2.11)$$

denotes the particle number operator.

The dynamics of ρ_t shall be determined by the time-dependent Hamiltonian

$$H(t) = H_0 + \eta V_\omega + \lambda W(t), \quad (2.12)$$

where the right hand side is defined as follows. The operator

$$H_0 := \int dp E(p) a_p^+ a_p \quad (2.13)$$

is the second quantization of the centered nearest neighbor Laplacian $(\Delta f)(x) = \sum_{|y-x|=1} f(y)$ on \mathbb{Z}^3 . The symbol of Δ is given by

$$E(p) = 2 \sum_{j=1}^3 \cos(2\pi p_j), \quad (2.14)$$

corresponding to the kinetic energy of a single electron. The interaction of the fermion gas with the static random background potential is described by the operator

$$V_\omega := \sum_{x \in \Lambda_L} \omega_x a_x^+ a_x. \quad (2.15)$$

We assume $\{\omega_x\}_{x \in \Lambda_L}$ to be a field of i.i.d. random variables which is centered, normalized, and Gaussian,

$$\mathbb{E}[\omega_x] = 0, \quad \mathbb{E}[\omega_x^2] = 1, \quad (2.16)$$

for $x \in \Lambda_L$. The small parameter $0 < \eta \ll 1$ controls the strength of the disorder.

The operator $W(t)$ accounts for fermion pair interactions in dynamical Hartree-Fock theory,

$$W(t) := \sum_{x,y \in \Lambda_L} v(x-y) \left[\mathbb{E}[\rho_t(a_x^+ a_x)] a_y^+ a_y - \mathbb{E}[\rho_t(a_y^+ a_y)] a_x^+ a_x \right]. \quad (2.17)$$

Here, v denotes a Λ_L -periodic pair potential, for which we assume that

$$\|\widehat{v}\|_{H^{3/2+\sigma}(\mathbb{T}^3)} < C \quad (2.18)$$

for $\sigma > 0$ arbitrary but fixed. Notably, we make no assumption on the sign of v . We note that the unknown quantity $\mathbb{E}[\rho_t(a_x^+ a_x)]$ appears in $W(t)$.

We are interested in the dynamics of the average of the pair correlation function,

$$\mathbb{E}[\rho_t(a_y^+ a_x)], \quad (2.19)$$

which is determined by the self-consistent nonlinear evolution equation

$$i\partial_t \mathbb{E}[\rho_t(a_x^+ a_y)] = \mathbb{E}[\rho_t([H(t), a_x^+ a_y])] \quad (2.20)$$

with initial condition $\rho_0(a_x^+ a_y)$. Its Fourier transform is diagonal in momentum space,

$$\mathbb{E}[\rho_t(a_p^+ a_q)] = \delta(p - q) \frac{1}{L^3} \mathbb{E}[\rho_t(a_p^+ a_p)], \quad (2.21)$$

because $\mathbb{E}[\rho_t(a_y^+ a_x)]$ is translation invariant. This follows from the homogeneity of the randomness.

We remark that for fermions,

$$0 \leq \frac{1}{L^3} \rho_0(a_p^+ a_p) \leq 1, \quad (2.22)$$

since $\|a_p^{(+)}\| = L^{d/2}$ in operator norm, $\forall p \in \Lambda_L^*$. The expected occupation density of the momentum p in the lattice Λ_L^* is given by

$$\mu_t(p) := \frac{1}{L^3} \mathbb{E}[\rho_t(a_p^+ a_p)]. \quad (2.23)$$

The dynamical Hartree-Fock interaction can be written as

$$\begin{aligned} W(t) &= \frac{1}{L^3} \rho_0(N) \left(\sum_x v(x) \right) N - \int_{\Lambda_L^*} dp (\hat{v} * \mu_t)(p) a_p^+ a_p \\ &=: W_{dir}(t) + W_{ex}(t) \end{aligned} \quad (2.24)$$

where, following standard terminology, $W_{dir}(t)$ denotes the direct, and $W_{ex}(t)$ the exchange term. It is clear that $H_\omega(t)$ is particle number conserving,

$$[H_\omega(t), N] = 0, \quad \forall t \in \mathbb{R}. \quad (2.25)$$

Since ρ_0 is translation invariant and number conserving, we conclude that whenever $[A, N] = 0$ holds for an operator A , it follows that

$$i\partial_t \rho_t(A) = \rho_t([H_{ex}(t), A]) \quad (2.26)$$

where the operator

$$H_{ex}(t) := H_0 + \eta V_\omega + \lambda W_{ex}(t) \quad (2.27)$$

contains only the exchange term of $W(t)$.

3. STATEMENT OF THE MAIN RESULTS

In order to determine the dynamics of the average momentum distribution function μ_t defined in (2.23), we consider

$$\begin{aligned} \int dp \overline{f(p)} g(p) \mu_t(p) &= \mathbb{E}[\rho_t(a^+(f)a(g))] \\ &= \mathbb{E}[\rho_0(\mathcal{U}_t^* a^+(f)a(g)\mathcal{U}_t)] \end{aligned} \quad (3.1)$$

for a translation invariant and particle number conserving initial state ρ_0 , where f and g are test functions. The linear operator \mathcal{U}_t denotes the unitary flow generated by $H_{ex}(t)$. It satisfies $\mathcal{U}_0 = \mathbf{1}$, and notably depends on μ_s , $s \in [0, t]$.

Accordingly, we make the key observation that (3.1) is a *fixed point equation* for μ_t , tested against f , g . The right hand side of (3.1) is a complicated nonlinear functional of μ_s which we will discuss in detail in Section 4.1.

We introduce macroscopic variables (T, X) , related to the microscopic variables (t, x) by

$$(T, X) = (\zeta t, \zeta x), \quad (3.2)$$

with $\zeta > 0$ a real parameter. We will study kinetic scaling limits associated to different scaling ratios between ζ , η and λ .

As stated in the introduction, the random potential has an average effect on the dynamics of μ_t by an amount proportional to its variance, $O(\eta^2 t)$, in the time interval $[0, t]$. Since the strength of the fermion pair interactions is $O(\lambda)$, both effects are comparable if $\lambda = O(\eta^2)$. This implies that the relevant scaling regimes of the system are determined by those addressed below, in Theorems 3.1, 3.2, and Theorem 3.4.

Theorem 3.1. *Assume that $\lambda \leq O(\eta^2)$. Then, for any fixed, finite $T > 0$, and any choice of test functions f , g ,*

$$\lim_{\eta \rightarrow 0} \lim_{L \rightarrow \infty} \mathbb{E}[\rho_{T/\eta^2}(a^+(f)a(g))] = \int dp \overline{f(p)} g(p) F_T(p) \quad (3.3)$$

holds, where $F_T(p)$ satisfies the linear Boltzmann equation

$$\partial_T F_T(p) = 2\pi \int du \delta(E(u) - E(p)) (F_T(u) - F_T(p)) \quad (3.4)$$

with initial condition $F_0 = \mu_0 \in H^{\frac{3}{2}+\sigma}(\mathbb{T}^3)$ for some $\sigma > 0$.

We note that the linear Boltzmann equation (3.4) can be explicitly solved. The solution is given by

$$F_T(p) = F_\infty(p) + e^{-Tm(p)} \frac{2\pi}{m(p)} \int du \delta(E(u) - E(p)) (F_0(u) - F_0(p)), \quad (3.5)$$

where

$$m(p) := 2\pi \int du \delta(E(u) - E(p)) \quad (3.6)$$

and

$$F_\infty(p) := \frac{2\pi}{m(p)} \int du \delta(E(u) - E(p)) F_0(u). \quad (3.7)$$

As an important example, we note that the following is obtained if the initial state ρ_0 is given by the Gibbs state for a non-interacting fermion gas (with inverse temperature β and chemical potential μ),

$$\rho_0(A) = \frac{1}{Z_{\beta,\mu}} \text{Tr}(e^{-\beta(T-\mu N)} A), \quad (3.8)$$

where $Z_{\beta,\mu} := \text{Tr}(e^{-\beta(T-\mu N)})$. The associated momentum distribution function is given by the Fermi-Dirac distribution

$$\lim_{L \rightarrow \infty} \frac{1}{L^3} \rho_0(a_p^+ a_p) = \frac{1}{1 + e^{\beta(E(p)-\mu)}}, \quad (3.9)$$

which is a *stationary solution* of the linear Boltzmann equation (3.4), for all $\beta > 0$. This result remains true in the zero temperature limit $\beta \rightarrow \infty$ where, in the weak sense,

$$\frac{1}{1 + e^{\beta(E(p)-\mu)}} \rightarrow \chi[E(p) < \mu] \quad (3.10)$$

(see also [10]).

In the case $\eta^2 = o_\lambda(1)$ and $T = \lambda t$, we find the following kinetic scaling limit.

Theorem 3.2. *Assume that $\eta^2 = O(\lambda^{1+\delta})$ for $\delta > 0$ arbitrary. Then, for any fixed, finite $T > 0$, and any choice of test functions f, g ,*

$$\lim_{\lambda \rightarrow 0} \lim_{L \rightarrow \infty} \mathbb{E}[\rho_{T/\lambda}(a^+(f) a(g))] = \int dp \overline{f(p)} g(p) F_T(p), \quad (3.11)$$

and

$$\partial_T F_T(p) = 0, \quad (3.12)$$

for $F_0 = \mu_0 \in H^{\frac{3}{2}+\sigma}(\mathbb{T}^3)$ for some $\sigma > 0$. Accordingly, $F_T = F_0$ is stationary.

Finally, we prove a partial result that highlights some interesting aspects about the problem of determining the kinetic scaling limit determined by $T = \eta^2 t$ and $\eta \rightarrow 0$, with λ small but independent of η . That is, we are considering, for $\lambda = O(1)$, the rescaled, formal fixed point equation

$$\begin{aligned} \int dp \overline{f(p)} g(p) \mu_{T/\eta^2}(p) &= \mathcal{G}^{(L)}[\mu_\bullet(\bullet); \eta; \lambda; T; f, g] \\ &:= \mathbb{E}[\rho_{T/\eta^2}(a^+(f) a(g))] \end{aligned} \quad (3.13)$$

for $\mu_\bullet(\bullet)$. The existence and uniqueness of solutions for this fixed point equation is currently an open problem. Below, we will make the assumption that there exist limiting stationary solutions, and determine their form under this hypothesis.

We base our discussion on the following hypotheses for the case $\lambda = O(1)$:

- (H1) There exist solutions $F^{(\eta)}(T) := \lim_{L \rightarrow \infty} \mu_{T/\eta^2}$ of (3.13), such that the limit $w - \lim_{\eta \rightarrow 0} F^{(\eta)}(T) =: F(T) = F(0)$ exists and is stationary.

(H2) The stationary fixed point solution in (H1) satisfies

$$\begin{aligned} F(T) &= \lim_{\eta \rightarrow 0} \lim_{L \rightarrow \infty} \mathcal{G}^{(L)}[F^{(\eta)}; \eta; \lambda; T; f, g] \\ &= \lim_{\eta \rightarrow 0} \lim_{L \rightarrow \infty} \mathcal{G}^{(L)}[F; \eta; \lambda; T; f, g]. \end{aligned} \quad (3.14)$$

The first equality sign here is equivalent to (H1), while the second equality sign accounts for the assumption that $F^{(\eta)}$ can be replaced by the limiting fixed point F before letting $\eta \rightarrow 0$, to produce the same result.

We remark that based on the analysis given in this paper, we are able to prove hypothesis (H2) if $F^{(\eta)} = F + O(\eta^2)$. Error bounds of order $O(\eta^2)$ require more precise estimates of "crossing" and "nesting" terms in the Feynman graph expansion than considered in this paper, but are available from [13, 14, 15, 16]. We will not further pursue this issue in the work at hand.

Proposition 3.3. *Let λ be small but independent of η , and assume that $F \in L^\infty(\mathbb{T}^3)$ independent of t . Then, the thermodynamic limit*

$$\mathcal{G}[F; \eta; \lambda; T; f, g] := \lim_{L \rightarrow \infty} \mathcal{G}^{(L)}[F; \eta; \lambda; T; f, g] \quad (3.15)$$

exists.

The proof of this proposition follows directly from results established in [8, 9, 10, 17], and will not be reiterated here.

Theorem 3.4. *Assume that $\lambda \leq O_\eta(1)$, and let*

$$\tilde{E}_\lambda(u) := E(u) - \lambda(\hat{v} * F)(u). \quad (3.16)$$

We assume that $F \in L^\infty(\mathbb{T}^3)$ admits the bounds

$$\sup_\alpha \int dp \frac{1}{|\tilde{E}_\lambda(q) - \alpha - i\epsilon|}, \sup_q \int d\alpha \frac{1}{|\tilde{E}_\lambda(q) - \alpha - i\epsilon|} \leq C \log \frac{1}{\epsilon}, \quad (3.17)$$

and

$$\begin{aligned} \sup_{\alpha_i} \sup_{u \in \mathbb{T}^3} \int dq dp \frac{1}{|\tilde{E}_\lambda(q) - \alpha_1 - i\epsilon|} \frac{1}{|\tilde{E}_\lambda(p) - \alpha_2 - i\epsilon|} \frac{1}{|\tilde{E}_\lambda(p \pm q + u) - \alpha_3 - i\epsilon|} \\ \leq \epsilon^{-b} \end{aligned} \quad (3.18)$$

for some $0 < b < 1$.

Then, F satisfies

$$\int dp \overline{f(p)} g(p) F(p) = \lim_{\eta \rightarrow 0} \mathcal{G}[F; \eta; \lambda; T; f, g], \quad (3.19)$$

independent of T , if and only if it satisfies

$$F(p) = \mu_0(p) = \frac{1}{\tilde{m}_\lambda(p)} \int du \delta(\tilde{E}_\lambda(u) - \tilde{E}_\lambda(p)) F(u), \quad (3.20)$$

where

$$\tilde{m}_\lambda(p) := 2\pi \int du \delta(\tilde{E}_\lambda(u) - \tilde{E}_\lambda(p)) \quad (3.21)$$

is the (normalized) measure of the level surface of \tilde{E}_λ for the value $\tilde{E}_\lambda(p)$.

Remark 3.5. *The following comments refer to Theorem 3.4.*

- (1) *The solution of (3.16) corresponds to a renormalized kinetic energy which is shifted by the average interaction energy for fermion pairs.*
- (2) *The fixed point equation (3.20) for F shows that the stationary kinetic limits of μ_t are concentrated and equidistributed on level surfaces of the renormalized kinetic energy function $\tilde{E}_\lambda(\cdot)$.*
- (3) *The bounds (3.17) and (3.18) correspond to the “crossing estimates” in [8, 17, 13, 21]. They ensure sufficient non-degeneracy of the renormalized energy level surfaces so that the Feynman graph expansions introduced below are convergent. However, they do not seem sufficient to prove hypothesis (H2) under the assumption that (H1) holds.*
- (4) *We note that if $\lambda \leq o_\eta(1)$, the stationary solutions found in Theorem 3.4 reduce to those of the linear Boltzmann equation derived in Theorem 3.1, see (3.7).*

4. FEYNMAN GRAPHS AND AMPLITUDES

In this section, we set up the Feynman graph expansions underlying our proofs of Theorems 3.1, 3.2, and Theorem 3.4.

4.1. Duhamel expansion. Let \mathcal{U}_t denote the unitary flow generated by $H_{ex}(t)$, determined by

$$i\partial_t \mathcal{U}_t = H_{ex}(t) \mathcal{U}_t \quad \text{and} \quad \mathcal{U}_0 = 1. \quad (4.1)$$

It then follows that

$$\rho_t(A) = \rho_0(\mathcal{U}_t^* A \mathcal{U}_t), \quad (4.2)$$

and that

$$\rho_t(a^+(f) a(g)) = \rho_0(a^+(f, t) a(g, t)). \quad (4.3)$$

The Heisenberg evolution of the creation- and annihilation operators is determined by

$$a(f, t) := \mathcal{U}_t^* a(f) \mathcal{U}_t, \quad (4.4)$$

with

$$a(f) = \int dp f(p) a_p, \quad a^+(f, t) = (a(f, t))^*. \quad (4.5)$$

It suffices to discuss the annihilation operators $a(f, t)$. Because $H_{ex}(t)$ is bilinear in a^+ and a , it follows that $a(f, t)$ is a linear superposition of annihilation operators. Therefore, there exists a function f_t such that

$$a(f, t) = a(f_t), \quad (4.6)$$

satisfying

$$\begin{aligned} i\partial_t a(f_t) &= [H_{ex}(t), a(f_t)] \\ &= \int dp f_t(p) E(p) a_p + \eta \int dp \int du f_t(p) \widehat{\omega}(u - p) a_u \\ &\quad - \lambda \int dp (\widehat{v} * \mu_t)(p) f_t(p) a_p, \end{aligned} \quad (4.7)$$

with initial condition

$$a(f, 0) = a(f_0) = a(f). \quad (4.8)$$

We conclude that f_t is the solution of the 1-particle random Schrödinger equation

$$i\partial_t f_t(p) = E(p) f_t(p) + \eta (\widehat{\omega} * f_t)(p) - \lambda (\widehat{v} * \mu_t)(p) f_t(p) \quad (4.9)$$

with initial condition

$$f_0 = f. \quad (4.10)$$

Here, $\widehat{\omega}(u) = \sum_x e^{2\pi i u x} \omega_x$, and v is the fermion pair interaction potential.

Noting that the Hamiltonian $H_{ex}(t)$ itself depends on the unknown quantity μ_t , we determine μ_t by writing the fixed point equation (3.1) in integral form, as an expansion in powers of η .

For arbitrary test functions f and g , we consider the pair correlation function

$$\begin{aligned}\rho_t(a^+(f) a(g)) &= \rho_0(a^+(f_t) a(g_t)) \\ &= \int dp dq \rho_0(a_p^+ a_q) \overline{f_t(p)} g_t(q) \\ &= \int dp J(p) \overline{f_t(p)} g_t(p),\end{aligned}\tag{4.11}$$

where the state ρ_t equals the one in the definition of μ_t , (2.23). Passing to the last line, we have used the momentum conservation condition

$$\rho_0(a_p^+ a_q) = J(p) \delta(p - q)\tag{4.12}$$

obtained from the translation invariance of the initial state ρ_0 , where

$$0 \leq J(p) = \frac{1}{L^3} \rho_0(a_p^+ a_p) \leq 1,\tag{4.13}$$

as noted before, in (2.22).

The solution f_t of (4.9), (4.10), satisfies the Duhamel (respectively, variation of constants) formula

$$f_t(p) = U_{0,t}(p) f(p) + i \eta \int_0^t ds U_{s,t}(p) (\widehat{\omega} * f_s)(p)\tag{4.14}$$

with

$$U_{s,t}(p) := e^{i \int_s^t ds' (E(p) - \lambda \kappa_{s'}(p))},\tag{4.15}$$

where we treat

$$\kappa_s(u) := (\widehat{v} * \mu_s)(u)\tag{4.16}$$

as an external (a priori bounded) source term. We note that $U_{0,t}(p)f(p)$ solves (4.9) for $\eta = 0$ (no random potential) with initial condition (4.10).

Let $N \in \mathbb{N}$, which remains to be optimized. The N -fold iterate of (4.14) produces the truncated Duhamel expansion with remainder term,

$$f_t = f_t^{(\leq N)} + f_t^{(> N)},\tag{4.17}$$

where

$$f_t^{(\leq N)} := \sum_{n=0}^N f_t^{(n)},\tag{4.18}$$

and $f_t^{(> N)}$ is the Duhamel remainder term of order N . We define

$$t_{-1} := 0, \quad t_j = s_0 + \cdots + s_j,\tag{4.19}$$

for $j = 0, \dots, n$, and

$$\mathcal{R}(k_0, \dots, k_n; z) := \int_{\mathbb{R}_+^{n+1}} ds_0 \cdots ds_n \left(\prod_{j=0}^n e^{-is_j(E(k_j) - z)} e^{i\lambda \int_{t_{j-1}}^{t_j} ds' \kappa_{s'}(k_j)} \right),\tag{4.20}$$

for $z \in \mathbb{C}$.

The n -th order term in the Duhamel expansion is given by

$$\begin{aligned} f_t^{(n)}(p) &:= (i\eta)^n \int_0^t dt_n \cdots \int_0^{t_2} dt_1 \int dk_0 \cdots dk_n \delta(p - k_0) \\ &\quad \left[\prod_{j=0}^n U_{t_{j-1}, t_j}(k_j) \right] \left[\prod_{\ell=1}^n \widehat{\omega}(k_\ell - k_{\ell-1}) \right] f(k_n). \end{aligned} \quad (4.21)$$

Expressed in terms of the time increments $s_j := t_j - t_{j-1}$,

$$\begin{aligned} f_t^{(n)}(p) &= (i\eta)^n \int ds_0 \cdots ds_n \delta(t - \sum_{j=0}^n s_j) \int dk_0 \cdots dk_n \delta(p - k_0) \\ &\quad \left[\prod_{j=0}^n e^{-i \int_{t_{j-1}}^{t_j} ds' (E(k_j) - \lambda \kappa_{s'}(k_j))} \right] \left[\prod_{\ell=1}^n \widehat{\omega}(k_\ell - k_{\ell-1}) \right] f(k_n). \end{aligned} \quad (4.22)$$

Expressing the delta distribution $\delta(t - \sum_{j=0}^n s_j)$ in terms of its Fourier transform, we find

$$\begin{aligned} f_t^{(n)}(p) &= (i\eta)^n e^{\epsilon t} \int d\alpha e^{-it\alpha} \int dk_0 \cdots dk_n \delta(p - k_0) \\ &\quad \mathcal{R}(k_0, \dots, k_n; \alpha + i\epsilon) \left[\prod_{j=1}^n \widehat{\omega}(k_j - k_{j-1}) \right] f(k_n). \end{aligned} \quad (4.23)$$

The above three equivalent expressions for $f_t^{(n)}(p)$ have different advantages which we will make use of.

The Duhamel remainder term of order N is given by

$$f_t^{(>N)} = i\eta \int_0^t ds \mathcal{U}_{s,t} V_\omega^{(1)} f_s^{(N)}. \quad (4.24)$$

We choose

$$\epsilon = \frac{1}{t} \quad (4.25)$$

so that the factor $e^{\epsilon t}$ in (4.23) remains bounded for all t .

Substituting the truncated Duhamel expansion for $a^+(f_t)$, $a(g_t)$ in (4.11), one obtains

$$\rho_t(a^+(f) a(g)) = \rho_0(a^+(f_t) a(g_t)) = \sum_{n, \tilde{n}=0}^{N+1} \rho_t^{(n, \tilde{n})}(f, g) \quad (4.26)$$

where

$$\rho_t^{(n, \tilde{n})}(f, g) := \rho_0(a^+(f_t^{(n)}) a(g_t^{(\tilde{n})})) \quad (4.27)$$

if $n, \tilde{n} \leq N$, and

$$\rho_t^{(n, N+1)}(f, g) := \rho_0(a^+(f_t^{(n)}) a(g_t^{(>N)})), \quad (4.28)$$

$$\rho_t^{(N+1, \tilde{n})}(f, g) := \rho_0(a^+(f_t^{(>N)}) a(g_t^{(\tilde{n})})) \quad (4.29)$$

if $n, \tilde{n} \leq N$. Moreover,

$$\rho_t^{(N+1, N+1)}(f, g) := \rho_0(a^+(f_t^{(>N)}) a(g_t^{(>N)})). \quad (4.30)$$

In particular, all Duhamel terms indexed by $n, \tilde{n} \leq N$ depend on $\widehat{\omega}$ like polynomials. Accordingly,

$$\begin{aligned} \mathbb{E}[\rho_t^{(n, \tilde{n})}(f, g)] &= \eta^{2\bar{n}} \sum_{\pi \in \Gamma_{n, \tilde{n}}} \int_0^t dt_q \cdots \int_0^{t_2} dt_1 \int_0^t d\tilde{t}_q \cdots \int_0^{\tilde{t}_2} d\tilde{t}_1 \\ &\quad \int du_0 \cdots du_{2\bar{n}+1} \overline{f(u_0)} g(u_{2\bar{n}+1}) J(u_n) \delta(u_n - u_{n+1}) \\ &\quad \left[\prod_{j=0}^n U_{t_{j-1}, t_j}(u_j) \right] \left[\prod_{j=n+1}^{2\bar{n}+1} \overline{U_{\tilde{t}_{j-1}, \tilde{t}_j}(u_j)} \right] \\ &\quad \mathbb{E} \left[\prod_{j=1}^n \widehat{\omega}(u_j - u_{j-1}) \prod_{j=n+2}^{2\bar{n}+1} \widehat{\omega}(u_j - u_{j-1}) \right] \end{aligned} \quad (4.31)$$

and using (4.23), this is equivalent to

$$\begin{aligned} \mathbb{E}[\rho_t^{(n, \tilde{n})}(f, g)] &= \eta^{2\bar{n}} e^{2\epsilon t} \sum_{\pi \in \Gamma_{n, \tilde{n}}} \int d\alpha d\tilde{\alpha} e^{it(\alpha - \tilde{\alpha})} \\ &\quad \int du_0 \cdots du_{2\bar{n}+1} \overline{f(u_0)} g(u_{2\bar{n}+1}) J(u_n) \delta(u_n - u_{n+1}) \\ &\quad \mathcal{R}(u_0, \dots, u_n; \alpha + i\epsilon) \mathcal{R}(u_{n+1}, \dots, u_{2\bar{n}+1}; \tilde{\alpha} - i\epsilon) \\ &\quad \mathbb{E} \left[\prod_{j=1}^n \widehat{\omega}(u_j - u_{j-1}) \prod_{j=n+2}^{2\bar{n}+1} \widehat{\omega}(u_j - u_{j-1}) \right] \end{aligned} \quad (4.32)$$

where $t_{-1}, \tilde{t}_{-1} := 0$ in (4.31).

4.2. Graph expansion. By assumption, $\{\omega_x\}$ is a centered, i.i.d., Gaussian random field. Accordingly, we may explicitly determine the correlations of the random potential in the expressions (4.31), (4.32). The expectation of any product of even degree $n \in 2\mathbb{N}$ is equal to the sum of all possible products of pair correlations of the same degree,

$$\begin{aligned} \mathbb{E} \left[\prod_{j=1}^n \widehat{\omega}(u_j - u_{j-1}) \right] \\ = \sum_{\text{pairings } (\ell_i, \ell'_i)} \prod_{i=1}^{\frac{n}{2}} \mathbb{E} \left[\widehat{\omega}(u_{\ell_i} - u_{\ell_{i-1}}) \widehat{\omega}(u_{\ell'_i} - u_{\ell'_{i-1}}) \right]. \end{aligned} \quad (4.33)$$

The sum extends over all possible pairings $(\ell_i, \ell'_i) \in \{1, \dots, n\}^2$ with $\ell_i \neq \ell'_i$, $i = 1, \dots, \frac{n}{2}$, where every element of $\{1, \dots, n\}$ appears in precisely one pairing. This expansion is often referred to as *Wick's theorem*. Expectations of products of $\widehat{\omega}$ of odd degree are identically zero.

To organize the terms in these sums of products of pair correlations, we introduce *Feynman graphs*. The set of Feynman graphs $\Gamma_{n, \tilde{n}}$, with $\bar{n} = n + \tilde{n} \in 2\mathbb{N}$, is given as follows; see also [8, 10, 17]:

- We consider two horizontal solid lines, which we refer to as *particle lines*, joined by a distinguished vertex which we refer to as the ρ_0 -vertex (corresponding to the term $\rho_0(a_{u_n}^+ a_{u_{n+1}})$). See Figure 1 for an example.
- On the line on its left, we introduce n vertices, and on the line on its right, we insert \tilde{n} vertices. We refer to those vertices as *interaction vertices*, and enumerate them from 1 to $2\bar{n}$ starting from the left.
- The edges between the interaction vertices are referred to as *propagator lines*. We label them by the momentum variables $u_0, \dots, u_{2\bar{n}+1}$, increasingly indexed starting from the left. To the j -th propagator line, we associate the propagator $U_{t_{j-1}, t_j}(u_j)$ if $0 \leq j \leq n$, and $\bar{U}_{\tilde{t}_{j-1}, \tilde{t}_j}(u_j)$ if $n+1 \leq j \leq 2\bar{n}+1$ (with reference to the expression (4.31)).
- To the ℓ -th interaction vertex (adjacent to the edges labeled by $u_{\ell-1}$ and u_ℓ), we associate the random potential $\hat{\omega}(u_\ell - u_{\ell-1})$, where $1 \leq \ell \leq 2\bar{n}+1$.

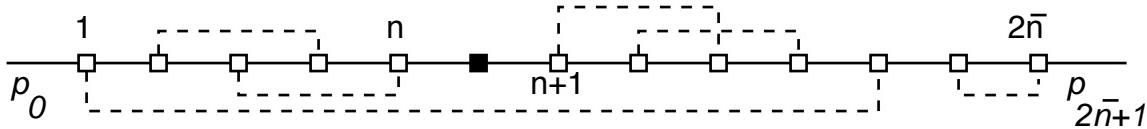


Figure 1. A contraction graph.

A *contraction graph* associated to the above pair of particle lines joined by the ρ_0 -vertex, and decorated by $n + \tilde{n}$ interaction vertices, is the graph obtained by pairwise connecting interaction vertices by *contraction lines*. We denote the set of all such contraction graphs by $\Gamma_{n,\tilde{n}}$; it contains

$$|\Gamma_{n,\tilde{n}}| = (2\bar{n}-1)(2\bar{n}-3) \cdots 3 \cdot 1 = \frac{(2\bar{n})!}{\bar{n}! 2^{\bar{n}}} = O(\bar{n}!) \quad (4.34)$$

elements.

If in a given graph $\pi \in \Gamma_{n,\tilde{n}}$, the ℓ -th and the ℓ' -th vertex are joined by a contraction line, we write

$$\ell \sim_\pi \ell', \quad (4.35)$$

and we associate the delta distribution

$$\delta(u_\ell - u_{\ell-1} - (u_{\ell'} - u_{\ell'-1})) = \mathbb{E}[\hat{\omega}(u_\ell - u_{\ell-1}) \hat{\omega}(u_{\ell'} - u_{\ell'-1})] \quad (4.36)$$

to this contraction line.

We classify Feynman graphs as follows; see also [8, 17]:

- A subgraph consisting of one propagator line adjacent to a pair of vertices ℓ and $\ell+1$, and a contraction line connecting them, i.e., $\ell \sim_\pi \ell+1$, where both $\ell, \ell+1$ are either $\leq n$ or $\geq n+1$, is called an *immediate recollision*.
- The graph $\pi \in \Gamma_{n,n}$ (i.e., $n = \tilde{n} = \bar{n}$) with $\ell \sim_\pi 2n - \ell$ for all $\ell = 1, \dots, n$, is called a *basic ladder* diagram. The contraction lines are called *rungs* of the ladder. We note that a rung contraction always has the form $\ell \sim_\pi \ell'$ with $\ell \leq n$ and $\ell' \geq n+1$. Moreover, in a basic ladder diagram one always has that if $\ell_1 \sim_\pi \ell'_1$ and $\ell_2 \sim_\pi \ell'_2$ with $\ell_1 < \ell_2$, then $\ell'_2 < \ell'_1$.

- A diagram $\pi \in \Gamma_{n,\tilde{n}}$ is called a *decorated ladder* if any contraction is either an immediate recollision, or a rung contraction $\ell_j \sim_\pi \ell'_j$ with $\ell_j \leq n$ and $\ell'_j \geq n$ for $j = 1, \dots, k$, and $\ell_1 < \dots < \ell_k, \ell'_1 > \dots > \ell'_k$. Evidently, a basic ladder diagram is the special case of a decorated ladder which contains no immediate recollisions (so that necessarily, $n = \tilde{n}$).
- A diagram $\pi \in \Gamma_{n,\tilde{n}}$ is called *crossing* if there is a pair of contractions $\ell \sim_\pi \ell', j \sim_\pi j'$, with $\ell < \ell'$ and $j < j'$, such that $\ell < j$.
- A diagram $\pi \in \Gamma_{n,\tilde{n}}$ is called *nesting* if there is a subdiagram with $\ell \sim_\pi \ell + 2k$, with $k \geq 1$, and either $\ell \geq n + 1$ or $\ell + 2k \leq n$, with $j \sim_\pi j + 1$ for $j = \ell + 1, \ell + 3, \dots, \ell + 2k - 1$. The latter corresponds to a progression of $k - 1$ immediate recollisions.

We note that any diagram that is not a decorated ladder contains at least a crossing or a nesting subdiagram.

4.3. Feynman amplitudes. To every Feynman graph $\pi \in \Gamma_{n,\tilde{n}}$ we associate its *Feynman amplitude*, as follows.

To start with, we recall from (4.20)

$$\begin{aligned} \mathcal{R}(u_0, \dots, u_n; \alpha + i\epsilon) \\ := \int_{\mathbb{R}_+^{n+1}} ds_0 \cdots ds_n \left(\prod_{j=0}^n e^{-is_j(E(k_j) - \alpha - i\epsilon)} e^{i\lambda \int_{t_{j-1}}^{t_j} ds' \kappa_{s'}(k_j)} \right), \end{aligned} \quad (4.37)$$

where $t_j = s_0 + \dots + s_j$ for $j > 0$, and $t_{-1} = 0$. Moreover, we recall that

$$\kappa_s(p) = (\hat{v} * \mu_s)(p) \quad (4.38)$$

where

$$0 \leq \mu_s(p) \leq 1 \quad (4.39)$$

holds uniformly in s and p . As a consequence,

$$\begin{aligned} \|\kappa_s\|_{L^\infty(\mathbb{T}^3)} &\leq \|\hat{v}\|_{L^1(\mathbb{T}^3)} = \int_{\mathbb{T}^3} dp \left| \sum_x v(x) e^{-2\pi ipx} \right| \\ &\leq \text{Vol}\{\mathbb{T}^3\} \|\langle \cdot \rangle^{3/2+\sigma} v\|_{\ell^2(\mathbb{Z}^3)} \|\langle \cdot \rangle^{-3/2-\sigma}\|_{\ell^2(\mathbb{Z}^3)} \\ &< C \|\hat{v}\|_{H^{3/2+\sigma}} \\ &< C' \end{aligned} \quad (4.40)$$

uniformly in $s \in \mathbb{R}_+$. Here, we have recalled the property (2.18) satisfied by the fermion pair interaction potential v , for constants C, C' that depend on $\sigma > 0$.

Given $\pi \in \Gamma_{n,\tilde{n}}$, we define

$$\delta_\pi(\{u_j\}_{j=0}^{2\tilde{n}+1}) := \prod_{\ell \sim_\pi \ell'} \delta(u_\ell - u_{\ell-1} - (u_{\ell'} - u_{\ell'-1})), \quad (4.41)$$

where every contraction line in π corresponds to one of the factors on the rhs, without any repetitions. Clearly, δ_π determines the momentum conservation conditions on the graph π .

Definition 4.1. *The Feynman amplitude associated to the graph $\pi \in \Gamma_{n,\tilde{n}}$ is defined by*

$$\begin{aligned} \text{Amp}_\pi(f, g; \epsilon; \eta) \\ := \eta^{2\tilde{n}} e^{2\epsilon t} \int d\alpha d\tilde{\alpha} e^{it(\alpha - \tilde{\alpha})} \\ \int du_0 \cdots du_{2\tilde{n}+1} \overline{f(u_0)} g(u_{2\tilde{n}+1}) J(u_n) \delta(u_n - u_{n+1}) \\ \delta_\pi(\{u_j\}_{j=0}^{2\tilde{n}+1}) \mathcal{R}(u_0, \dots, u_n; \alpha + i\epsilon) \mathcal{R}(u_{n+1}, \dots, u_{2\tilde{n}+1}; \tilde{\alpha} - i\epsilon). \end{aligned} \quad (4.42)$$

Our choice of ϵ will be $\epsilon = \frac{1}{t}$.

Since $\{\omega_x\}$ are i.i.d. centered Gaussian,

$$\mathbb{E}\left[\prod \widehat{\omega}(u_\ell - u_{\ell-1})\right] = \sum_{\pi \in \Gamma_{n,\tilde{n}}} \prod_{i \sim \pi j} \mathbb{E}\left[\widehat{\omega}(u_i - u_{i-1}) \widehat{\omega}(u_j - u_{j-1})\right] \quad (4.43)$$

equals the sum of all possible products of pair correlations

$$\mathbb{E}[\widehat{\omega}(u) \widehat{\omega}(u')] = \delta(u + u') \quad (4.44)$$

(Wick's theorem). Accordingly,

$$\mathbb{E}[\rho_t^{(n,\tilde{n})}(f, g)] = \sum_{\pi \in \Gamma_{n,\tilde{n}}} \text{Amp}_\pi(f, g; \epsilon; \eta) \quad (4.45)$$

is the sum of Feynman amplitudes of all Feynman graphs $\pi \in \Gamma_{n,\tilde{n}}$.

As a consequence of translation invariance of ρ_0 , we have that

$$\rho_0(a_{u_n}^+ a_{u_{n+1}}^-) = J(u_n) \delta(u_n - u_{n+1}), \quad (4.46)$$

as we recall from (4.12). Moreover, translation invariance also implies overall momentum conservation, that is,

$$u_0 - u_{2\tilde{n}+1} = 0, \quad (4.47)$$

which one easily verifies by summing up the arguments of all delta distributions.

Accordingly, we arrive at the expansion

$$\begin{aligned} \mathbb{E}[\rho_t(f, g)] &= \sum_{n, \tilde{n}=0}^{N+1} \sum_{\pi \in \Gamma_{n,\tilde{n}}} \text{Amp}_\pi(f, g; \epsilon; \eta) \\ &+ \sum_{n=0}^N (\mathbb{E}[\rho_t^{(n,N+1)}(f, g)] + \mathbb{E}[\rho_t^{(N+1,n)}(f, g)]) \\ &+ \mathbb{E}[\rho_t^{(N+1,N+1)}(f, g)] \end{aligned} \quad (4.48)$$

where the first term on the rhs is entirely expressed in terms of Feynman graphs and Feynman amplitudes. The terms on the second and third line on the r.h.s. involve the Duhamel remainder term, and will be shown only to contribute to a small error.

5. PROOF OF THEOREM 3.1: I. BOUNDS ON ERROR TERMS

In order to prove the Boltzmann limit stated in Theorem 3.1, we separate the main terms in the expression (4.48) from the error terms. We subsequently show that the main terms converge to a solution of the Boltzmann equation, while the error terms tend to zero. The Feynman graphs associated to the main term correspond to those appearing in the works [17, 8, 10]. However, many aspects of the approach developed in those works (for the weakly disordered Anderson model, which is linear) are not suitable for the problem at hand. The main issue is the presence of the phase $\lambda \int_{\tau}^{\tau+s} \kappa_{s'}(u) ds'$ in (4.37), which depends on the unknown quantity μ_t itself. In this section, we introduce a main tool, given in Lemmata 5.1 and 5.2, that enables us to control the nonlinear self-interaction of the fermion field.

In a first step, we prove an estimate that will serve as a substitute for resolvent estimates. The latter were abundantly used in [17, 8], but due to the nonlinear self-interactions of the fermion field, they are not available here.

Lemma 5.1. *Let $\epsilon = \frac{1}{t} \ll 1$. Then, there exists a constant $C < \infty$ independent of η, λ, ϵ such that*

$$|\mathcal{R}(u_0, \dots, u_n; \alpha + i\epsilon)| \leq C^{n+1} \left(1 + \frac{\lambda}{\epsilon}\right)^{n+1} \prod_{j=0}^n \frac{1}{|E(u_j) - \alpha| + \epsilon} \quad (5.1)$$

for all $n \in \mathbb{N}$.

Lemma 5.1 follows from Lemma 5.2 below. We note that in our proof below, we allow for at most one integration by parts with respect to the time variable s , which requires no smoothness assumption $\kappa_s(u)$. Iterating integration by parts with respect to s would easily produce the asserted result, but under the assumption that κ_s is C^n -smooth in s . However, we will only rely on the a priori boundedness of $\kappa_s(u)$, which is a consequence of the Fermi statistics satisfied by the quantum field, and will not assume any smoothness with respect to s .

Lemma 5.2. *Assume (4.40). Then, uniformly in $\tau \geq 0$,*

$$\left| \int_{\mathbb{R}^+} ds e^{-is(E(u) - \alpha - i\epsilon)} e^{-i\lambda \int_{\tau}^{\tau+s} \kappa_{s'}(u) ds'} \right| < \left(1 + \frac{\lambda}{\epsilon}\right) \frac{C}{|E(u) - \alpha| + \epsilon}, \quad (5.2)$$

where $E(u)$ is the symbol of the nearest neighbor Laplacian on \mathbb{Z}^3 .

Proof. We define

$$\bar{\kappa}_{t,t+s}(u) := \frac{1}{s} \int_t^{t+s} ds' \kappa_{s'}(u). \quad (5.3)$$

Clearly, $|\bar{\kappa}_{t,t+s}(u)| < C_0$, uniformly in t and $s \geq 0$.

The integral on the left hand side of (5.2) can be written as

$$\int_{\mathbb{R}^+} ds e^{-is(E(u) - \alpha + \lambda \bar{\kappa}_{t,t+s}(u))} e^{-\epsilon s}. \quad (5.4)$$

To estimate it, we split \mathbb{R}_+ into disjoint intervals

$$I_j := [j\zeta, (j+1)\zeta), \quad j \in \mathbb{N}_0 \quad (5.5)$$

of length

$$\zeta := \frac{\pi}{|E(u) - \alpha|}. \quad (5.6)$$

We find

$$(5.4) = \sum_{j \in 2\mathbb{N}_0} \int_{I_j} ds \left(e^{-is(E(u)-\alpha+\lambda\bar{\kappa}_{t,t+s}(u))} e^{-\epsilon s} + e^{-i(s+\zeta)(E(u)-\alpha+\lambda\bar{\kappa}_{t,t+s+\zeta}(u))} e^{-\epsilon(s+\zeta)} \right), \quad (5.7)$$

where the second term in the bracket accounts for the integrals over I_j with j odd.

Evidently, $e^{-i\zeta(E(u)-\alpha)} = e^{\mp i\pi} = -1$. Therefore, we get, for j fixed,

$$\begin{aligned} & \int_{I_j} ds \left(e^{-is(E(u)-\alpha+\lambda\bar{\kappa}_{t,t+s}(u))} e^{-\epsilon s} + e^{-i(s+\zeta)(E(u)-\alpha+\lambda\bar{\kappa}_{t,t+s+\zeta}(u))} e^{-\epsilon(s+\zeta)} \right) \\ &= \int_{I_j} ds e^{-is(E(u)-\alpha+\lambda\bar{\kappa}_{t,t+s}(u))} (e^{-\epsilon s} - e^{-\epsilon(s+\zeta)}) \end{aligned} \quad (5.8)$$

$$+ \int_{I_j} ds e^{-is(E(u)-\alpha)} e^{-\epsilon(s+\zeta)} (e^{-i\lambda s \bar{\kappa}_{t,t+s}(u)} - e^{-i\lambda(s+\zeta)\bar{\kappa}_{t,t+s}(u)}) \quad (5.9)$$

$$+ \int_{I_j} ds e^{-is(E(u)-\alpha)} e^{-\epsilon(s+\zeta)} (e^{-i\lambda(s+\zeta)\bar{\kappa}_{t,t+s}(u)} - e^{-i\lambda(s+\zeta)\bar{\kappa}_{t,t+s+\zeta}(u)}). \quad (5.10)$$

Clearly,

$$\sum_{j \in 2\mathbb{N}_0} |(5.8)| < \int_{\mathbb{R}_+} ds e^{-\epsilon s} \epsilon \zeta = \frac{\pi}{|E(u) - \alpha|}, \quad (5.11)$$

and

$$\sum_{j \in 2\mathbb{N}_0} |(5.9)| < \int_{\mathbb{R}_+} ds e^{-\epsilon s} \lambda \zeta = \frac{\lambda}{\epsilon} \frac{\pi}{|E(u) - \alpha|}. \quad (5.12)$$

On the other hand, we observe that for $s_1 < s_2$,

$$\bar{\kappa}_{t,t+s_2}(u) - \bar{\kappa}_{t,t+s_1}(u) = \left(\frac{1}{s_2} - \frac{1}{s_1} \right) \int_t^{t+s_2} ds' \kappa_{s'}(u) \quad (5.13)$$

$$+ \frac{1}{s_1} \left(\int_t^{t+s_2} - \int_t^{t+s_1} \right) ds' \kappa_{s'}(u). \quad (5.14)$$

Since $|\kappa_{s'}(u)| < C_0$ uniformly in s' , we immediately obtain

$$|\bar{\kappa}_{t,t+s_2}(u) - \bar{\kappa}_{t,t+s_1}(u)| < C \frac{s_2 - s_1}{s_1}, \quad (5.15)$$

so that in particular,

$$|\bar{\kappa}_{t,t+\zeta+s}(u) - \bar{\kappa}_{t,t+s}(u)| < C \frac{\zeta}{s}. \quad (5.16)$$

Thus, we conclude that

$$\begin{aligned} \sum_{j \in 2\mathbb{N}_0} (5.10) &\leq C \int_{\mathbb{R}_+} ds \lambda \zeta \frac{(s + \zeta)}{s} e^{-\epsilon(s + \zeta)} \\ &\leq C \frac{\lambda}{\epsilon} \frac{\pi}{|E(u) - \alpha|}. \end{aligned} \quad (5.17)$$

This proves that for $|E(u) - \alpha| > 0$,

$$|(5.7)| < \frac{C}{|E(u) - \alpha|}, \quad (5.18)$$

under the assumption that $\lambda = O(\epsilon)$.

If $|E(u) - \alpha| \leq \epsilon$, then the trivial bound

$$|(5.7)| < \int_{\mathbb{R}_+} ds e^{-\epsilon s} < \frac{C}{\epsilon} \quad (5.19)$$

is better, which ignores phase cancellations, so that in conclusion,

$$|(5.7)| < \frac{C}{|E(u) - \alpha| + \epsilon}, \quad (5.20)$$

as claimed. \square

We may now prove Lemma 5.1.

Proof. We consider

$$\int_{\mathbb{R}_+^{n+1}} ds_0 \cdots ds_n \prod_{j=0}^n e^{-i\Phi(s_j, u_j, t_{j-1})} e^{-\epsilon s_j} \quad (5.21)$$

where

$$\Phi(s, u, t) := s(E(u) - \alpha + \lambda \bar{\kappa}_{t, t+s}(u)) \quad (5.22)$$

and

$$t_j := s_0 + \cdots + s_j, \quad t_{-1} := 0. \quad (5.23)$$

Then, subdividing \mathbb{R}_+ into intervals I_ℓ as before,

$$\begin{aligned} |(5.21)| &\leq \sum_{\ell_0, \dots, \ell_n \in 2\mathbb{N}_0} \int_{I_{\ell_0}} ds_0 \left| e^{-i\Phi(s_0, u_0, t_{-1})} e^{-\epsilon s_j} - e^{-i\Phi(s_0 + \zeta, u_0, t_{-1})} e^{-\epsilon(s_0 + \zeta)} \right| \\ &\quad \left\{ \cdots \int_{I_{\ell_n}} ds_n \left| e^{-i\Phi(s_n, u_n, t_{n-1})} e^{-\epsilon s_n} - e^{-i\Phi(s_n + \zeta, u_n, t_{n-1})} e^{-\epsilon(s_n + \zeta)} \right| \right\} \end{aligned} \quad (5.24)$$

we may use $L^1 - L^\infty$ bounds in s_j to get

$$\begin{aligned}
(5.24) \leq & \sum_{\ell_0, \dots, \ell_n \in 2\mathbb{N}_0} \prod_{j=0}^n \sup_{s_0, \dots, s_{j-1}} \int_{I_{\ell_j}} ds_j \left| e^{-i\Phi(s_j, u_j, t_{j-1})} e^{-\epsilon s_j} \right. \\
& \quad \left. - e^{-i\Phi(s_j + \zeta, u_j, t_{j-1})} e^{-\epsilon(s_j + \zeta)} \right| \\
\leq & \prod_{j=0}^n C(\epsilon + \lambda) \frac{1}{|E(u_j) - \alpha| + \epsilon} \int_{\mathbb{R}_+} ds_j e^{-\epsilon s_j} \\
\leq & \prod_{j=0}^n \frac{C}{|E(u_j) - \alpha| + \epsilon} \tag{5.25}
\end{aligned}$$

by application of Lemma 5.2, recalling the assumption that $\lambda \leq O(\epsilon)$. In particular, the fact has been proven here that the bound proven in Lemma 5.2 holds uniformly with respect to t . \square

Moreover, we prove the following stationary phase estimate.

Lemma 5.3. *Assume that $f, \widehat{v} \in H^{3/2+\sigma}(\mathbb{T}^3)$ for some $\sigma > 0$, and that $\lambda \leq O(\eta^2)$. Then, for $0 < s < t = O(\eta^{-2})$, and uniformly in $\tau \in \mathbb{R}$,*

$$\begin{aligned}
& \left| \int_{\mathbb{T}^3} du e^{-is(E(u) - \alpha - i\epsilon)} e^{i\lambda \int_\tau^{\tau+s} \kappa_s(u)} f(u) \right| \\
& \quad < C(\sigma) \langle s \rangle^{-3/2} \|\widehat{v}\|_{H^{3/2+\sigma}(\mathbb{T}^3)} \|f\|_{H^{3/2+\sigma}(\mathbb{T}^3)} \tag{5.26}
\end{aligned}$$

where $\kappa_s = \mu_s * \widetilde{v}$.

Proof. Let

$$\begin{aligned}
g_s(u) &:= e^{-is(E(u) - \alpha - i\epsilon)}, \\
h_s(u) &:= e^{i\lambda \int_t^{t+s} ds' \kappa_{s'}(u)}. \tag{5.27}
\end{aligned}$$

Clearly, the left hand side of (5.26) satisfies

$$\begin{aligned}
\left| \int_{\mathbb{T}^3} du g_s(u) h_s(u) \right| &\leq \|\langle x \rangle^{-3/2-\sigma} g_s^\vee\|_{\ell^2(\mathbb{Z}^3)} \|\langle x \rangle^{3/2+\sigma} (f h_s)^\vee\|_{\ell^2(\mathbb{Z}^3)} \\
&< \frac{C'}{\sigma} \|g_s^\vee\|_{\ell^\infty(\mathbb{Z}^3)} \|f h_s\|_{H^{3/2+\sigma}(\mathbb{T}^3)} \tag{5.28}
\end{aligned}$$

$$< \frac{C}{\sigma} \langle s \rangle^{-3/2} \|f\|_{H^{3/2+\sigma}(\mathbb{T}^3)} \|h_s\|_{H^{3/2+\sigma}(\mathbb{T}^3)}, \tag{5.29}$$

where we used the fact that $H^\alpha(\mathbb{T}^d)$, with $\alpha > \frac{d}{2}$, is a Banach algebra.

With $\lambda s \leq O(1)$,

$$\|h_s\|_{H^{3/2+\sigma}(\mathbb{T}^3)} < C \|\kappa_s\|_{H^{3/2+\sigma}(\mathbb{T}^3)} \leq C \|\mu_s\|_{L^1(\mathbb{T}^3)} \|\widehat{v}\|_{H^{3/2+\sigma}(\mathbb{T}^3)} \tag{5.30}$$

where $\|\mu_s\|_{L^1(\mathbb{T}^3)} \leq \|\mu_s\|_{L^\infty(\mathbb{T}^3)} < c$ uniformly in s . Accordingly, (5.26) follows. \square

5.1. Feynman diagrams contributing to the error term. In this section, we combine Lemmata 5.1 and 5.2, and the results of Section 6.5 below, to prove that the amplitudes of all Feynman graphs that include a crossing or nesting subdiagram contribute only to a small error. In addition, we show that all terms involving the Duhamel remainder term likewise contribute to a small error.

We point out that in our approach, the estimates provided in Lemmata 5.1 and 5.2, and in Section 6.5, require first integrating out all time variables s_j , and subsequently integrating out the momenta u_j in (4.42) and (4.37).

This makes it possible to straightforwardly adopt estimates on crossing, nesting, and remainder terms from [8, 9, 10, 17]. Accordingly, our discussion in this section can be kept short.

5.1.1. Crossing and nesting diagrams. We defer the discussion of nesting diagrams to Section 6.5 below because the necessary ingredients are introduced only in later sections. We shall here anticipate the result from the analysis given there.

Using Lemmata 5.1 and 5.2, we may infer from the analysis presented in [8, 9, 10, 17] that the Feynman amplitude of any graph $\pi \in \Gamma_{n,\tilde{n}}$ containing a crossing or nesting subdiagram is bounded by

$$\begin{aligned} & \lim_{L \rightarrow \infty} |\text{Amp}_\pi(f, g; \epsilon; \eta)| \\ & \leq \|f\|_2 \|g\|_2 \|J\|_\infty \epsilon^{1/5} (\log \frac{1}{\epsilon})^4 (c\eta^2 \epsilon^{-1} \log \frac{1}{\epsilon})^{\bar{n}}, \end{aligned} \quad (5.31)$$

where $2\bar{n} = n + \tilde{n} \in 2\mathbb{N}$. The thermodynamic limit $L \rightarrow \infty$ and the upper bound in (5.31) are obtained in the same manner as in [8, 9, 10, 17] where we refer for details. We do not repeat the discussion here, and instead refer to those references.

Let

$$\Gamma_{n,\tilde{n}}^{c-n} \subset \Gamma_{n,\tilde{n}} \quad (5.32)$$

denote the subset of Feynman graphs of crossing or nesting type, and

$$\Gamma_{2\bar{n}}^{c-n} := \bigcup_{n+\tilde{n}=2\bar{n}} \Gamma_{n,\tilde{n}}^{c-n}, \quad (5.33)$$

with cardinality $|\Gamma_{2\bar{n}}^{c-n}| \leq 2^{\bar{n}} \bar{n}!$.

Accordingly, summing over all graphs with a crossing or nesting subdiagram,

$$\begin{aligned} & \sum_{1 \leq \bar{n} \leq N} \sum_{\pi \in \Gamma_{2\bar{n}}^{c-n}} \lim_{L \rightarrow \infty} |\text{Amp}_\pi(f, g; \epsilon; \eta)| \\ & < \|f\|_2 \|g\|_2 \|J\|_\infty \epsilon^{1/5} (\log \frac{1}{\epsilon})^4 (c\eta^2 \epsilon^{-1} N \log \frac{1}{\epsilon})^N, \end{aligned} \quad (5.34)$$

where we replaced the factorials by $N! < N^N$. Since f, g are of Schwartz class, $\|f\|_2, \|g\|_2 < C$.

Moreover,

$$\|J\|_\infty \leq 1, \quad (5.35)$$

from (2.22). As a concrete example,

$$0 \leq J(p) = (1 + e^{\beta(E(p) - \mu)})^{-1} \leq 1, \quad (5.36)$$

for the Gibbs state of a free Fermi gas, at inverse temperature $0 \leq \beta \leq \infty$.

5.1.2. Remainder term. Using Lemmata 5.1 and 5.2, and following [10] (which uses results in [8, 9, 17]), we straightforwardly find the following bounds on the contributions of the Duhamel remainder term.

If at least one of the indices n, \tilde{n} equals $N + 1$, one obtains

$$\begin{aligned} & \lim_{L \rightarrow \infty} \|\mathbb{E}[\rho^{(n, \tilde{n})}(f, g)]\| \\ & \leq \|f\|_2 \|g\|_2 \|J\|_\infty \left[\frac{N^2 \kappa^2}{(N!)^{1/2}} \max\{c \epsilon^{-1} \eta^2, (c \epsilon^{-1} \eta^2)^N\} \right. \\ & \quad \left. + (N^2 \kappa^2 \epsilon^{1/5} + \epsilon^{-2} \kappa^{-N}) (\log \frac{1}{\epsilon})^4 (c \eta^2 \epsilon^{-1} N \log \frac{1}{\epsilon})^{8N} \right], \end{aligned} \quad (5.37)$$

(again using $(4N)! < (4N)^{4N}$) where the constant $1 \ll \kappa \ll t$ remains to be optimized. The first term on the right hand side of (5.37) bounds the contribution from all basic ladder diagrams contained in the Duhamel expanded remainder term. A detailed discussion of an analogous result, and more details, can be found in [8, 9, 10, 17].

5.1.3. Choice of parameters and error bounds. The kinetic scaling limit asserted in Theorem 3.1 is determined by

$$t = \frac{1}{\epsilon} = \frac{T}{\eta^2} \quad (5.38)$$

where t denotes the microscopic, and T the macroscopic time variable. Similarly as in [8, 9, 10, 17], we choose

$$\begin{aligned} N = N(\epsilon) &= \frac{\log \frac{1}{\epsilon}}{10 \log \log \frac{1}{\epsilon}} \\ \kappa &= (\log \frac{1}{\epsilon})^{15}, \end{aligned} \quad (5.39)$$

so that

$$\begin{aligned} \epsilon^{-1/11} &< N! &< \epsilon^{-1/10} \\ \kappa^N &> \epsilon^{-3/2}. \end{aligned} \quad (5.40)$$

One can easily verify that, accordingly, for any choice of $T > 0$ finite and fixed,

$$(5.34) < \eta^{1/15} \quad (5.41)$$

$$(5.37) < \eta^{1/4}, \quad (5.42)$$

for η sufficiently small.

In conclusion, we have proven the following overall bound on the amplitudes of all Feynman graphs that we attribute to the error term in the expansion (4.48).

Proposition 5.4. *For the choice of parameters (5.39),*

$$(5.34) + (5.37) < \eta^{\frac{1}{20}} \quad (5.43)$$

holds, for any choice of $T > 0$ finite and fixed. The l.h.s. of (5.43) contains the sum of Feynman amplitudes associated to all diagrams containing crossing and/or nesting subgraphs. It also contains the sum of all terms that depend on the remainder term of the Duhamel expansion.

6. PROOF OF THEOREM 3.1: II. RESUMMATION OF MAIN TERMS

In this section, we discuss the main terms in the expression (4.48), which are associated to the set of decorated ladder diagrams. This is the complement of the set of Feynman graphs containing crossing and/or nesting subgraphs studied in the previous section. We prove that the sum of amplitudes of all decorated ladders converges to a solution of the linear Boltzmann equation (3.4), for $\lambda \leq O(\eta^2)$. We remark that because of the presence of the nonlinear self-interaction of the fermion field, our analysis here is much more involved than the analogous part in [8, 17].

We let $\Gamma_{n,\tilde{n}}^{(lad)} \subset \Gamma_{n,\tilde{n}}$ denote the subset of all decorated ladders based on $n + \tilde{n}$ vertices. Our goal is to prove that in the given kinetic scaling limit, the sum of amplitudes associated to decorated ladder graphs converges to a solution of the linear Boltzmann equation (3.4).

6.1. Outline of the proof. Our discussion is organized as follows:

Step 1. In Sections 6.2 and 6.3, we consider the Feynman amplitudes associated to propagator lines decorated with an arbitrary number of immediate recollisions. The sum of all such terms produces a renormalized propagator (renormalization in one-loop approximation, according to standard terminology in quantum field theory). We prove that the leading term corresponds to a multiplicative renormalization of the basic propagator $U_{t_a,t_b}(u)$; see (4.15) for its definition.

Step 2. In Section 6.4, we discuss the Feynman amplitudes of basic ladder diagrams (that is, the propagators between rungs do not contain any immediate recollisions). We prove that, in the kinetic scaling limit, the Feynman amplitudes of these graphs determine a transport equation obtained from omitting the loss term on the rhs of the linear Boltzmann equation (3.4). To prove that the error terms thus obtained are small, we apply various stationary phase estimates to the propagators $U_{t_a,t_b}(u)$.

Step 3. In Section 6.6, we combine the results obtained in the previous two steps, and show that by decorating the basic ladder diagrams with immediate recollisions, the full Boltzmann equation (3.4) is obtained in the kinetic scaling limit.

As a key result of Step 2, we obtain that the dominant part of the Feynman amplitudes is in fact independent of λ in the regime $\lambda = O(\eta^2)$. This is a consequence of cancellations due to the translation invariance of the system.

6.2. Immediate recollisions. According to step 1 outlined above, we study the amplitude of a progression of m immediate recollisions.

Lemma 6.1. *For any connected interval $I \subset \mathbb{R}_+$ of length $|I| \leq O(t)$,*

$$\left| \int_{\mathbb{T}^3} du \int_I ds e^{-i \int_t^{t+s} ds' (E(u) - \alpha - i\epsilon \lambda \kappa_{s'}(u))} \right. \\ \left. - \int_{\mathbb{T}^3} du \int_I ds e^{-is(E(u) - \alpha - i\epsilon)} \right| < C \lambda |I|^{1/2}, \quad (6.1)$$

and

$$\int_I ds \left| \int_{\mathbb{T}^3} du e^{-i \int_t^{t+s} ds' (E(u) - \lambda \kappa_{s'}(u))} \right| \leq C (1 + \lambda |I|^{1/2}) \quad (6.2)$$

uniformly in $t \geq 0$.

Proof. To prove (6.1), we write

$$\int_I ds \int_{\mathbb{T}^3} du e^{-is(E(u)-\alpha-i\epsilon)} e^{i\lambda \int_t^{t+s} ds' \kappa_{s'}(u)} = (I) + (II) \quad (6.3)$$

where

$$(I) := \int_I ds \int_{\mathbb{T}^3} du e^{-is(E(u)-\alpha-i\epsilon)} \quad (6.4)$$

and

$$(II) := i \int_0^\lambda d\lambda' \int_I ds \int_{\mathbb{T}^3} du e^{-is(E(u)-\alpha-i\epsilon)} \int_t^{t+s} ds' \kappa_{s'}(u) e^{i\lambda' \int_t^{t+s'} ds'' \kappa_{s''}(u)}, \quad (6.5)$$

from differentiating and integrating with respect to the parameter λ . To estimate (II), we introduce a smooth partition of unity,

$$\int_{\mathbb{T}^3} du = \sum_{j=1}^8 \int_{\mathbb{T}^3} du \chi_j(u) \quad (6.6)$$

where the smooth, \mathbb{T}^3 -periodic bump function χ_j is centered around the j -th critical point of $E(u)$, which is a real analytic, perfect Morse function. The properties (2.18) satisfied by \widehat{v} also hold for $\kappa_s = \widehat{v} * \mu_s$ (see (5.30)). Lemma 5.3, applied on the support of each χ_j , implies that

$$\begin{aligned} & \left| \int_{\mathbb{T}^3} du e^{-is(E(u)-\alpha-i\epsilon)} \int_t^{t+s} ds' \kappa_{s'}(u) e^{i\lambda' \int_t^{t+s'} ds'' \kappa_{s''}(u)} \right| \\ & \leq C \langle s \rangle^{-3/2} e^{-\epsilon s} \left\| \int_t^{t+s} ds' \chi_j(u) \kappa_{s'}(u) e^{i\lambda' \int_t^{t+s'} ds'' \kappa_{s''}(u)} \right\|_{H^{3/2+\sigma}(\mathbb{T}^3)} \\ & \leq C \langle s \rangle^{-3/2} s e^{-\epsilon s} \end{aligned} \quad (6.7)$$

for any finite $\sigma > 0$, as long as $\lambda \leq O(\epsilon)$, and $s \leq \epsilon^{-1}$ (where $\langle s \rangle = (1 + s^2)^{1/2}$). It thus follows that

$$|(II)| \leq \lambda \int_I ds \langle s \rangle^{-3/2} s e^{-\epsilon s} = O(\lambda |I|^{1/2}), \quad (6.8)$$

which proves (6.1). In the same manner,

$$\left| \int_{\mathbb{T}^3} du e^{-i \int_t^{t+s} ds' (E(u) - \lambda \kappa_{s'}(u))} \right| < C \langle s \rangle^{-3/2} (1 + \lambda s), \quad (6.9)$$

implies (6.2). \square

6.3. Renormalized propagators. Next, we resum progressions of immediate recollisions of arbitrary length, leading to a propagator renormalization.

Let $m \in \mathbb{N}$, and $S := \{1, \dots, 2m+1\}$. Moreover, let $0 \leq t_a < t_b \leq t$. We consider the integral

$$\begin{aligned} U_{t_a, t_b}^{(m)}(u) := & (-\eta^2)^m \int_{\mathbb{R}_+^{2m+1}} ds_1 \cdots ds_{2m+1} \delta(t_b - t_a - \sum s_j) \\ & \int_{(\mathbb{T}^3)^{2m}} du_2 \cdots du_{2m+1} \prod_{j=1}^m \delta(u_{2j+1} - u_{2j-1}) \\ & \prod_{\ell=1}^{2m+1} e^{-i \int_{t_{\ell-1}}^{t_\ell} ds'(E(u_\ell) - \lambda \kappa_{s'}(u_\ell))} \end{aligned} \quad (6.10)$$

corresponding to the Feynman graph given by a progression of m immediate recollisions. Integrating out the product of m delta distributions,

$$\begin{aligned} U_{t_a, t_b}^{(m)}(u) = & U_{t_a, t_b}^{(0)}(u) \\ & (-\eta^2)^m \int_{\mathbb{R}_+^{2m+1}} ds_1 \cdots ds_{2m+1} \delta(t_b - t_a - \sum s_j) \\ & \prod_{j \in S \cap 2\mathbb{N}} \int_{\mathbb{T}^3} du' e^{-i \int_{t_{j-1}}^{t_j} ds'(E(u') - E(u) - \lambda(\kappa_{s'}(u') - \kappa_{s'}(u)))} \end{aligned} \quad (6.11)$$

where

$$t_j = t_a + s_1 + \cdots + s_j, \quad (6.12)$$

for $j = 1, \dots, 2m+1$, and where

$$U_{t_a, t_b}^{(0)}(u) = e^{-i \int_{t_a}^{t_b} ds'(E(u) - \lambda \kappa_{s'}(u))} \quad (6.13)$$

by definition of $U_{t_a, t_b}^{(m)}(u)$.

In the following lemma, we determine the Feynman amplitude of a progression of m immediate recollisions, and extract the dominant part. We identify the latter as a multiplicative renormalization of the free evolution term (6.13).

Lemma 6.2. *Assume that $\lambda \leq O(\eta^2)$ and $0 \leq t_a < t_b \leq t$. Then, for all $m \in \mathbb{N}$ and $\nu = O(\eta^2)$,*

$$U_{t_a, t_b}^{(m)}(u) = U_{t_a, t_b}^{(main;m)}(u) + \Delta U_{t_a, t_b}^{(m)}(u) \quad (6.14)$$

where

$$U_{t_a, t_b}^{(main;m)}(u) := \frac{1}{m!} \left(-\eta^2 (t_b - t_a) \int du' \frac{1}{E(u') - E(u) - i\nu} \right)^m U_{t_a, t_b}^{(0)}(u) \quad (6.15)$$

and

$$\| \Delta U_{t_a, t_b}^{(m)} \|_{L^\infty(\mathbb{T}^3)} \leq (m \lambda \nu^{-1/2} + \nu^{1/2}) (c \eta^2 \nu^{-1})^m. \quad (6.16)$$

In particular, $\Delta U_{t_a, t_b}^{(0)}(u) = 0$ in the case $m = 0$.

Proof. We represent the delta distribution in (6.11) by

$$\begin{aligned}\delta(t_b - t_a - \sum s_j) &= e^{\nu(t_b - t_a)} e^{-\nu \sum s_j} \delta(t_b - t_a - \sum s_j) \\ &= e^{\nu(t_b - t_a)} e^{-\nu \sum s_j} \frac{1}{2\pi} \int_{\mathbb{R}} d\gamma e^{-i(t_b - t_a - \sum s_j)\gamma} \\ &= e^{\nu(t_b - t_a)} \frac{1}{2\pi} \int_{\mathbb{R}} d\gamma e^{-i(t_b - t_a)\gamma} e^{i(\sum s_j)(\gamma + i\nu)},\end{aligned}\quad (6.17)$$

where we will pick $\nu = \epsilon = \frac{1}{t} = \frac{\eta^2}{T}$ so that $(t_b - t_a)\nu \leq \nu t = 1$ and in particular, $e^{\nu(t_b - t_a)} \leq e$ uniformly in η . In order to keep track of the origin of these two parameters, we will continue to notationally distinguish ν and ϵ .

From (6.11) and (6.17),

$$\begin{aligned}U_{t_a, t_b}^{(m)}(u) &= e^{-i \int_{t_a}^{t_b} ds'(E(u) - \lambda \kappa_{s'}(u) + i\nu)} \\ &\quad \frac{1}{2\pi} \int_{\mathbb{R}} d\gamma e^{-i\gamma(t_b - t_a)} (-\eta^2)^m \int_{\mathbb{R}_{+}^{2m+1}} ds_1 \cdots ds_{2m+1} \prod_{\ell \in S \cap 2\mathbb{N}-1} e^{is_{\ell}(\gamma + i\nu)} \\ &\quad \prod_{j \in S \cap 2\mathbb{N}} \int_{\mathbb{T}^3} du' e^{-i \int_{t_{j-1}}^{t_j} ds'(E(u') - E(u) - \gamma - i\nu - \lambda(\kappa_{s'}(u') - \kappa_{s'}(u)))}.\end{aligned}\quad (6.18)$$

where according to the above definitions, $t_j = t_{j-1} + s_j$. Our goal is to prove that

$$\begin{aligned}U_{t_a, t_b}^{(m)}(u) &= e^{-i \int_{t_a}^{t_b} ds'(E(u) - \lambda \kappa_{s'}(u) + i\nu)} \\ &\quad \left\{ \frac{1}{2\pi} \int d\gamma e^{-i\gamma(t_b - t_a)} \left(\frac{i}{\gamma + i\nu} \right)^{m+1} \left(\int du' \frac{-i\eta^2}{E(u') - E(u) - \gamma - i\nu} \right)^m \right. \\ &\quad \left. + \text{err}_1 \right\} \\ &= e^{-i \int_{t_a}^{t_b} ds'(E(u) - \lambda \kappa_{s'}(u))} \left\{ e^{\nu(t_b - t_a)} \right. \\ &\quad \left. \frac{1}{2\pi} \int d\gamma e^{-i\gamma(t_b - t_a)} \left(\frac{i}{\gamma + i\nu} \right)^{m+1} \left(\int du' \frac{-i\eta^2}{E(u') - E(u) - i\nu} \right)^m \right. \\ &\quad \left. + \text{err}_1 + \text{err}_2 \right\} \\ &= ie^{-i \int_{t_a}^{t_b} ds'(E(u) - \lambda \kappa_{s'}(u))} \left\{ \frac{(t_b - t_a)^m}{m!} \left(-\eta^2 \int du' \frac{1}{E(u') - E(u) - i\nu} \right)^m \right. \\ &\quad \left. + \text{err}_1 + \text{err}_2 \right\},\end{aligned}\quad (6.19)$$

where we claim that

$$|\text{err}_1| < m \lambda \nu^{-1/2} (c \eta^2 \nu^{-1})^m \quad (6.20)$$

and

$$|\text{err}_2| < \nu^{1/2} (c \eta^2 \nu^{-1})^m. \quad (6.21)$$

Clearly, this implies the asserted bound.

We first prove (6.20). To this end, we will use that $\|\kappa_s\|_{H^{3/2+}(\mathbb{T}^3)} < c$ uniformly in s , see (5.30), and the fact that the kinetic energy $E(u)$ is a real analytic, perfect

Morse function on \mathbb{T}^3 . A stationary phase estimate similar to the one applied in the proof of Lemma 5.3 yields

$$\begin{aligned} & \left| \int_{\mathbb{T}^3} du' e^{-i \int_{t_{j-1}}^{t_j} ds' (E(u') - E(u) - \gamma - i\nu - \lambda(\kappa_{s'}(u') - \kappa_{s'}(u)))} \right. \\ & \quad \left. - \int_{\mathbb{T}^3} du' e^{-i(t_j - t_{j-1})(E(u') - E(u) - \gamma - i\nu)} \right| \\ & \leq \int_0^\lambda d\lambda' \left| \int du' \left(\int_{t_{j-1}}^{t_j} ds'' \kappa_{s''}(u') \right) \right. \\ & \quad \left. e^{-i \int_{t_{j-1}}^{t_j} ds' (E(u') - E(u) - \gamma - i\nu - \lambda'(\kappa_{s'}(u') - \kappa_{s'}(u)))} \right| \\ & < C \lambda s_j \langle s_j \rangle^{-3/2}, \end{aligned} \quad (6.22)$$

where $t_j = t_{j-1} + s_j$. Thus,

$$\begin{aligned} |\text{err}_1| & < C \eta^{2m} \int_{\mathbb{R}_+^{2m+1}} ds_1 \cdots ds_{2m+1} \delta(t_b - t_a - \sum s_j) \\ & \quad \sum_{\ell \in S \cap 2\mathbb{N}} \lambda \langle s_\ell \rangle^{-1/2} \prod_{j \in S \cap 2\mathbb{N}; j \neq \ell} \langle s_j \rangle^{-3/2} \\ & < C m \lambda \nu^{-1/2} (C \nu^{-1} \eta^2)^m. \end{aligned} \quad (6.23)$$

This implies (6.20).

To estimate err_2 , we use

$$\begin{aligned} |\text{err}_2| & = \left| \frac{1}{2\pi} \left(\int_{|\gamma|<\nu} + \int_{|\gamma|\geq\nu} \right) d\gamma e^{-i\gamma(t_b - t_a)} \left(\frac{i}{\gamma + i\nu} \right)^{m+1} \right. \\ & \quad \left[\left(\int du' \frac{i\eta^2}{E(u') - E(u) - \gamma - i\nu} \right)^m \right. \\ & \quad \left. \left. - \left(\int du' \frac{i\eta^2}{E(u') - E(u) - i\nu} \right)^m \right] \right| \\ & \leq C \eta^{2m} \int_{|\gamma|\geq\nu} d\gamma \left(\frac{1}{|\gamma| + \nu} \right)^{m+1} \\ & \quad \left(2 \sup_{\gamma \in \mathbb{R}} \sup_{E(u) \in [-6, 6]} \left| \int du' \frac{1}{E(u') - E(u) - \gamma - i\nu} \right|^m \right) \\ & + C \eta^{2m} \int_{|\gamma|<\nu} d\gamma \left(\frac{1}{|\gamma| + \nu} \right)^{m+1} \\ & \quad \sum_{j=1}^m \binom{m}{j} (\Delta(\nu))^j \left| \int du' \frac{1}{E(u') - E(u) - i\nu} \right|^{m-j} \end{aligned} \quad (6.24)$$

where

$$\begin{aligned} \Delta(\nu) & := \sup_{|\gamma|<\nu} \sup_{E(u) \in [-6, 6]} \left| \int du' \frac{1}{E(u') - E(u) - \gamma - i\nu} \right. \\ & \quad \left. - \int du' \frac{1}{E(u') - E(u) - i\nu} \right|. \end{aligned} \quad (6.25)$$

We prove below that

$$\Delta(\nu) < C \nu^{1/2} \quad (6.26)$$

and

$$\sup_{\gamma \in \mathbb{R}} \sup_{E(u) \in [-6, 6]} \left| \int du' \frac{1}{E(u') - E(u) - i\nu} \right| < C. \quad (6.27)$$

Recalling that $\sum_{j=1}^m \binom{m}{j} = 2^m - 1$, this implies that

$$(6.24) < (C \nu^{-1/2} \eta^2)^m + \nu^{1/2} (C \nu^{-1} \eta^2)^m. \quad (6.28)$$

Choosing $\nu = \epsilon = O(\eta^2)$, we arrive at the asserted bound for $|\text{err}_2|$.

To prove (6.26), we use that, from a stationary phase estimate,

$$\begin{aligned} & \left| \int du' \frac{1}{E(u') - E(u) - \gamma - i\nu} - \int du' \frac{1}{E(u') - E(u) - i\nu} \right| \\ &= \left| \int_0^\gamma d\gamma' \int du' \left(\frac{1}{E(u') - E(u) - \gamma - i\nu} \right)^2 \right| \\ &= \left| \int_0^\gamma d\gamma' \int_0^\infty ds_1 \int_{s_1}^\infty ds_2 \int du' e^{-is_2(E(u') - E(u) - \gamma - i\nu)} \right| \\ &\leq C |\gamma| \int_0^\infty ds_1 \int_{s_1}^\infty ds_2 \langle s_2 \rangle^{-3/2} e^{-\nu s_2} \\ &\leq C \nu \nu^{-1/2}, \end{aligned} \quad (6.29)$$

since $|\gamma| < \nu$, and since $E(\cdot)$ is a real analytic, perfect Morse function on \mathbb{T}^3 , as noted before. This implies (6.26).

On the other hand,

$$\begin{aligned} & \left| \int du' \frac{1}{E(u') - E(u) - \gamma - i\nu} \right| \\ &= \left| \int_0^\infty ds \int du' e^{-is(E(u') - E(u) - \gamma - i\nu)} \right| \\ &< C' \int_0^\infty ds \langle s \rangle^{-3/2} e^{-\nu s} \\ &< C \end{aligned} \quad (6.30)$$

for all $\gamma \in \mathbb{R}$, and in particular for $|\gamma| \geq \nu$. This implies (6.27). \square

6.4. Sum of basic ladders. Following step 2 described at the beginning of Section 6, we now determine the kinetic scaling limit of the sum of all Feynman amplitudes associated to basic ladder graphs. In combination with the propagator renormalization addressed in the previous section, this will allow us to complete the proof of the Boltzmann limit asserted in Theorem 3.1.

The Feynman amplitude of a single basic ladder graph with q rungs is given by

$$\begin{aligned} \mathcal{U}_t^{(basic;q)}(J; f, g) &:= (-\eta^2)^q \int_0^t dt_q \cdots \int_0^{t_2} dt_1 \int_0^t d\tilde{t}_q \cdots \int_0^{\tilde{t}_2} d\tilde{t}_1 \\ &\int du_0 \cdots du_q J(u_q) \overline{f(u_0)} g(u_0) \\ &U_{t_q, t}^{(0)}(u_q) \overline{U_{\tilde{t}_q, t}^{(0)}(u_q)} \cdots U_{t_1, t_2}^{(0)}(u_1) \overline{U_{\tilde{t}_1, \tilde{t}_2}^{(0)}(u_1)} U_{0, t_1}^{(0)}(u_0) \overline{U_{0, \tilde{t}_1}^{(0)}(u_0)} \end{aligned} \quad (6.31)$$

where the definition of $U_{t_{j-1}, t_j}^{(0)}(u)$ is given in (6.13).

The following intermediate result will be important for the derivation of the full transport equations in the next section.

Proposition 6.3. *Assume that $\lambda = o(\eta)$, and $M(\eta) \in \mathbb{N}$ with $M(\eta) \leq O(\frac{\log \frac{1}{\eta}}{\log \log \frac{1}{\eta}})$ and $\lim_{\eta \rightarrow 0} M(\eta) = \infty$. Then, for any fixed, finite $T > 0$,*

$$F_T^{(basic)}(J; f, g) := \lim_{\eta \rightarrow 0} \sum_{q=0}^{M(\eta)} \mathcal{U}_{T/\eta^2}^{(basic;q)}(J; f, g) \quad (6.32)$$

exists, and

$$F_T^{(basic)}(J; f, g) = \int du \overline{f(u)} g(u) F_T^{(basic)}(u) \quad (6.33)$$

where $F_T^{(basic)}(u)$ satisfies

$$\partial_T F_T^{(basic)}(u) = \int du' \delta(E(u) - E(u')) F_T^{(basic)}(u'), \quad (6.34)$$

with initial condition $F_0(u) = J(u)$.

This proposition is an immediate consequence of the following lemma.

Lemma 6.4. *Assume that $\lambda = o(\eta)$. Then,*

$$\begin{aligned} & \left| \mathcal{U}_t^{(basic;q)}(J; f, g) - \mathcal{U}_t^{(basic-main;q)}(J; f, g) \right| \\ & \quad < C q \lambda t^{1/2} (\eta^2 t \log \frac{1}{\eta})^{q-1} + C q \eta (\eta^2 t \log \frac{1}{\eta})^{q-1}, \end{aligned} \quad (6.35)$$

where $\mathcal{U}_t^{(basic-main;q)}(J; f, g)$ is defined in (6.47) below, and

$$F_T^{(basic;q)}(J; f, g) := \lim_{\eta \rightarrow 0} \mathcal{U}_{T/\eta^2}^{(basic-main;q)}(J; f, g) \quad (6.36)$$

exists for any fixed, finite $T > 0$. In particular,

$$F_T^{(basic;q)}(J; f, g) = \int du \overline{f(u)} g(u) F_T^{(basic;q)}(u) \quad (6.37)$$

where $F_T^{(basic;q)}(u)$ satisfies

$$\partial_T F_T^{(basic;q)}(u) = \int du' \delta(E(u) - E(u')) F_T^{(basic;q-1)}(u') \quad (6.38)$$

and $F_0^{(basic;q)}(u) = 0$ if $q \geq 1$, and $F_0^{(basic;0)}(u) = J(u)$.

Proof. First, we write

$$U_{t_{j-1}, t_j}^{(0)}(u_j) = U_{0, t_j}^{(0)}(u_j) \overline{U_{0, t_{j-1}}^{(0)}(u_j)}, \quad (6.39)$$

so that

$$\begin{aligned} & U_{t_{j-1}, t_j}^{(0)}(u_j) U_{t_{j-2}, t_{j-1}}^{(0)}(u_{j-1}) \\ & = U_{0, t_j}^{(0)}(u_j) \left(U_{0, t_{j-1}}^{(0)}(u_{j-1}) \overline{U_{0, t_{j-1}}^{(0)}(u_j)} \right) \overline{U_{0, t_{j-2}}^{(0)}(u_{j-1})} \end{aligned} \quad (6.40)$$

and

$$\begin{aligned} & U_{t_{j-1}, t_j}^{(0)}(u_j) U_{t_{j-2}, t_{j-1}}^{(0)}(u_{j-1}) \overline{U_{\tilde{t}_{j-1}, \tilde{t}_j}^{(0)}(u_j)} \overline{U_{t_{j-2}, \tilde{t}_{j-1}}^{(0)}(u_{j-1})} \\ &= U_{\tilde{t}_j, t_j}^{(0)}(u_j) \left(U_{\tilde{t}_{j-1}, t_{j-1}}^{(0)}(u_{j-1}) \overline{U_{t_{j-1}, t_{j-1}}^{(0)}(u_j)} \right) \overline{U_{t_{j-2}, t_{j-2}}^{(0)}(u_{j-1})} \end{aligned} \quad (6.41)$$

where $U_{t,t'}^{(0)}(u) = \overline{U_{t',t}^{(0)}(u)}$ (due to $\int_t^{t'} = -\int_{t'}^t$ on the rhs of (6.13)) independently of $t > t'$ or $t < t'$. Accordingly, we find

$$\begin{aligned} & \mathcal{U}_t^{(basic;q)}(J; f, g) \\ &:= (-\eta^2)^q \int_0^t dt_q \int_0^{t_q} dt_{q-1} \cdots \int_0^{t_2} dt_1 \int_0^t d\tilde{t}_q \int_0^{\tilde{t}_q} d\tilde{t}_{q-1} \cdots \int_0^{\tilde{t}_2} d\tilde{t}_1 \\ & \quad \int du_0 \cdots du_q J(u_q) \overline{f(u_0)} g(u_0) \\ & \quad U_{t,t}^{(0)}(u_q) \left(\overline{U_{t_q, t_q}^{(0)}(u_q)} U_{t_q, t_q}^{(0)}(u_{q-1}) \right) \cdots \left(\overline{U_{\tilde{t}_1, t_1}^{(0)}(u_1)} U_{\tilde{t}_1, t_1}^{(0)}(u_0) \right) \end{aligned} \quad (6.42)$$

where, evidently, $U_{t,t}^{(0)}(u_q) = 1$. Let $\delta_u(u') := \delta(u' - u)$ so that

$$\mathcal{U}_t^{(basic;q)}(J; f, g) = \int du J(u) \mathcal{U}_t^{(basic;q)}(\delta_u; f, g). \quad (6.43)$$

Our proof comprises the following main steps.

Step (1) First, we verify that $\mathcal{U}_t^{(basic;q)}(J; f, g)$ satisfies the following approximate recursive identity,

$$\begin{aligned} & \mathcal{U}_t^{(basic;q)}(J; f, g) \\ &= -\eta^2 \int du_q J(u_q) \int du_{q-1} \int_0^t dt_q \int_0^t d\tilde{t}_q \left(\overline{U_{\tilde{t}_q, t_q}^{(0)}(u_q)} U_{\tilde{t}_q, t_q}^{(0)}(u_{q-1}) \right) \\ & \quad \mathcal{U}_{t_q}^{(basic;q-1)}(\delta_{u_{q-1}}; f, g) \\ & \quad + O\left(t^{-1/2} \frac{(C\eta^2 t)^q}{(q-2)!}\right). \end{aligned} \quad (6.44)$$

We note that the main term in (6.44) differs from (6.42) only by the upper integration boundary for the variable \tilde{t}_{q-1} , which is replaced by $\tilde{t}_{q-1} \rightarrow t_q$.

Step (2) Next, we prove that for $\lambda = o(\eta)$, the nonlinear self-interaction of the fermion field only contributes to a small error,

$$\begin{aligned} & \mathcal{U}_t^{(basic;q)}(J; f, g) \\ &= -\eta^2 \int du_q J(u_q) \int du_{q-1} \int_0^t dt_q \int_{-t}^t ds \exp\left(-is(E(u_q) - E(u_{q-1}))\right) \\ & \quad \mathcal{U}_{t_q}^{(basic;q-1)}(\delta_{u_{q-1}}; f, g) \\ & \quad + O(\lambda t^{1/2} (\eta^2 t \log \frac{1}{\eta})^{q-1}) + O(\eta (\eta^2 t \log \frac{1}{\eta})^{q-1}). \end{aligned} \quad (6.45)$$

Step (3) Iterating (6.45), one gets

$$\begin{aligned} \mathcal{U}_t^{(basic;q)}(J; f, g) &= \mathcal{U}_t^{(basic-main;q)}(J; f, g) \\ &+ O(q \lambda t^{1/2} (\eta^2 t \log \frac{1}{\eta})^{q-1}) + O(q \eta (\eta^2 t \log \frac{1}{\eta})^{q-1}) \end{aligned} \quad (6.46)$$

where

$$\begin{aligned} \mathcal{U}_t^{(basic-main;q)}(J; f, g) &:= (-\eta^2)^q \int du_q J(u_q) \int du_{q-1} \cdots du_0 \overline{f(u_0)} g(u_0) \\ &\int_0^t dt_q \cdots \int_0^{t_2} dt_1 \int_{-t}^t ds_q \cdots \int_{-t_2}^{t_2} ds_1 \prod_{j=1}^q \exp(-is_j(E(u_j) - E(u_{j-1}))) . \end{aligned} \quad (6.47)$$

Introducing the variables $T = \eta^2 t$ and $T_j = \eta^2 t_j$,

$$\begin{aligned} \mathcal{U}_{T/\eta^2}^{(basic-main;q)}(J; f, g) &= (-1)^q \int du_q J(u_q) \int du_{q-1} \cdots du_0 \overline{f(u_0)} g(u_0) \int_0^T dT_q \cdots \int_0^{T_2} dT_1 \\ &\int_{-T/\eta^2}^{T/\eta^2} ds_q \cdots \int_{t_2/\eta^2}^{t_2/\eta^2} ds_1 \prod_{j=1}^q \exp(-is_j(E(u_j) - E(u_{j-1}))) . \end{aligned} \quad (6.48)$$

For every $T_{j+1} > 0$,

$$\int_{T_{j+1}/\eta^2}^{T_{j+1}/\eta^2} ds_j \exp(-is_j(E(u_j) - E(u_{j-1}))) \longrightarrow \delta(E(u_j) - E(u_{j-1})) \quad (6.49)$$

weakly in the limit $\eta \rightarrow 0$. Therefore,

$$\begin{aligned} \lim_{\eta \rightarrow 0} \mathcal{U}_{T/\eta^2}^{(basic-main;q)}(J; f, g) &= \frac{(-T)^q}{q!} \int du_0 \cdots du_q J(u_q) \overline{f(u_0)} g(u_0) \prod_{j=1}^q \delta(E(u_j) - E(u_{j-1})) . \end{aligned} \quad (6.50)$$

It is easy to check that (6.50) has the properties asserted in the lemma. For more details on the limit $\eta \rightarrow 0$, we refer to [17].

It thus remains to prove steps (1) and (2).

Proof of step (1). The difference between $\mathcal{U}_t^{(basic;q)}(J; f, g)$ and the first term on the rhs of (6.44) is given by

$$\begin{aligned} & \Delta\mathcal{U}_t^{(basic;q)}(J; f, g) \\ &:= (\text{First term on rhs of (6.44)}) - \mathcal{U}_t^{(basic;q)}(J; f, g) \\ &= (-\eta^2)^q \int_0^t dt_q \int_0^{t_q} dt_{q-1} \cdots \int_0^{t_2} dt_1 \int_0^{t_q} d\tilde{t}_q \int_{\tilde{t}_q}^{t_q} d\tilde{t}_{q-1} \cdots \int_0^{\tilde{t}_2} d\tilde{t}_1 \\ &\quad \int du_0 \cdots du_q J(u_q) \overline{f(u_0)} g(u_0) \\ &\quad U_{t,t}^{(0)}(u_q) \left(\overline{U_{\tilde{t}_q,t_q}^{(0)}(u_q)} U_{\tilde{t}_q,t_q}^{(0)}(u_{q-1}) \right) \cdots \left(\overline{U_{\tilde{t}_1,t_1}^{(0)}(u_1)} U_{\tilde{t}_1,t_1}^{(0)}(u_0) \right). \end{aligned} \quad (6.51)$$

The only difference between this expression and the expression (6.42) consists of the integration boundaries for the variable \tilde{t}_{q-1} .

We use a stationary phase argument similarly as in Lemma 5.3 to bound the integral in u_q , which yields

$$\left| \int du_q J(u_q) \overline{U_{\tilde{t}_q,t_q}^{(0)}(u_q)} \right| < C \| J \|_{H^{\frac{3}{2}+\sigma}(\mathbb{T}^3)} \langle t_q - \tilde{t}_q \rangle^{-3/2}, \quad (6.52)$$

for some $\sigma > 0$, and similarly,

$$\left| \int du_0 \overline{f(u_0)} g(u_0) U_{\tilde{t}_1,t_1}^{(0)}(u_0) \right| < C \| f \|_{H^{\frac{3}{2}+\sigma}(\mathbb{T}^3)} \| g \|_{H^{\frac{3}{2}+\sigma}(\mathbb{T}^3)} \langle t_1 - \tilde{t}_1 \rangle^{-3/2} \quad (6.53)$$

recalling that $H^{\frac{3}{2}+\sigma}(\mathbb{T}^3)$ is an algebra.

For the integrals in u_j with $j = 2, \dots, q-1$,

$$\left| \int du_j U_{\tilde{t}_j,t_j}^{(0)}(u_j) \overline{U_{\tilde{t}_{j-1},t_{j-1}}^{(0)}(u_j)} \right| < C \langle t_j - \tilde{t}_j - (t_{j-1} - \tilde{t}_{j-1}) \rangle^{-3/2}. \quad (6.54)$$

Accordingly, writing

$$B_{J,f,g} := \| J \|_{H^{\frac{3}{2}+\sigma}(\mathbb{T}^3)} \| f \|_{H^{\frac{3}{2}+\sigma}(\mathbb{T}^3)} \| g \|_{H^{\frac{3}{2}+\sigma}(\mathbb{T}^3)} \quad (6.55)$$

we find

$$\begin{aligned} & \left| \Delta\mathcal{U}_t^{(basic;q)}(J; f, g) \right| \\ & \leq C^q B_{J,f,g} \eta^{2q} \int_0^t dt_q \int_0^t d\tilde{t}_q \frac{1}{\langle t_q - \tilde{t}_q \rangle^{3/2}} \\ & \quad \int_0^{t_q} dt_{q-1} \int_{\tilde{t}_q}^{t_q} d\tilde{t}_{q-1} \frac{1}{\langle t_q - \tilde{t}_q - (t_{q-1} - \tilde{t}_{q-1}) \rangle^{3/2}} \\ & \quad \int_0^{t_{q-1}} dt_{q-2} \cdots \int_0^{t_2} dt_1 \int_{\tilde{t}_q}^{t_{q-1}} d\tilde{t}_{q-2} \cdots \int_0^{\tilde{t}_2} d\tilde{t}_1 \\ & \quad \frac{1}{\langle t_1 - \tilde{t}_1 \rangle^{3/2}} \prod_{j=2}^{q-1} \frac{1}{\langle t_j - \tilde{t}_j - (t_{j-1} - \tilde{t}_{j-1}) \rangle^{3/2}}. \end{aligned} \quad (6.56)$$

To bound the integrals in \tilde{t}_j with $j = 1, \dots, q-1$, we use

$$\begin{aligned}
& \int_0^{\tilde{t}_{q-1}} d\tilde{t}_{q-2} \frac{1}{\langle t_{q-1} - \tilde{t}_{q-1} - (t_{q-2} - \tilde{t}_{q-2}) \rangle^{3/2}} \cdots \cdots \\
& < \int_{\mathbb{R}} d\tilde{t}_{q-2} \frac{1}{\langle t_{q-1} - \tilde{t}_{q-1} - (t_{q-2} - \tilde{t}_{q-2}) \rangle^{3/2}} \cdots \cdots \\
& < \left(\int_{\mathbb{R}} dt' \frac{1}{\langle t' \rangle^{3/2}} \right)^{q-3} \sup_{t' \in \mathbb{R}} \int_{\mathbb{R}} d\tilde{t}_1 \frac{1}{\langle t_2 - \tilde{t}_2 - (t_1 - \tilde{t}_1) \rangle^{3/2}} \frac{1}{\langle t_1 - \tilde{t}_1 \rangle^{3/2}} \\
& < C^{q-2}, \tag{6.57}
\end{aligned}$$

where we first majorized the expression by extending all integration intervals to \mathbb{R} , and subsequently translated all variables $\tilde{t}_j \rightarrow \tilde{t}_j + t_j$. The integrals in t_j with $j = 1, \dots, q-2$ are easily seen to be bounded by $\frac{t^{q-2}}{(q-2)!}$. We thus find

$$\begin{aligned}
& \left| \Delta \mathcal{U}_t^{(basic;q)}(J; f, g) \right| \\
& \leq C^q B_{J,f,g} \eta^4 \int_0^t dt_q \sup_{t_q} \left\{ \int_0^t d\tilde{t}_q \frac{1}{\langle t_q - \tilde{t}_q \rangle^{3/2}} \right. \\
& \quad \left. \int_{\tilde{t}_q}^{t_q} d\tilde{t}_{q-1} \sup_{t_q, \tilde{t}_q, \tilde{t}_{q-1}} \left[\int_0^{t_q} dt_{q-1} \frac{1}{\langle t_q - \tilde{t}_q - (t_{q-1} - \tilde{t}_{q-1}) \rangle^{3/2}} \right] \right\} \frac{(\eta^2 t)^{q-2}}{(q-2)!} \\
& \leq C^q B_{J,f,g} \eta^4 \int_0^t dt_q \sup_{t_q} \left\{ \int_0^t d\tilde{t}_q \frac{1}{\langle t_q - \tilde{t}_q \rangle^{1/2}} \right\} \frac{(\eta^2 t)^{q-2}}{(q-2)!} \\
& \leq C^q B_{J,f,g} \eta^4 t^{1/2} \frac{(\eta^2 t)^{q-2}}{(q-2)!}, \tag{6.58}
\end{aligned}$$

using that

$$\int_{\tilde{t}_q}^{t_q} d\tilde{t}_{q-1} \sup_{t_q, \tilde{t}_q, \tilde{t}_{q-1}} \left[\int_0^{t_q} dt_{q-1} \frac{1}{\langle t_q - \tilde{t}_q - (t_{q-1} - \tilde{t}_{q-1}) \rangle^{3/2}} \right] \leq \langle t_q - \tilde{t}_q \rangle. \tag{6.59}$$

Consequently,

$$\left| \Delta \mathcal{U}_t^{(basic;q)}(J; f, g) \right| < B_{J,f,g} t^{-1/2} \frac{(C \eta^2 t)^q}{(q-2)!} \tag{6.60}$$

follows, as claimed.

Proof of step (2). Similarly as in Lemma 6.1,

$$\begin{aligned}
\left(\overline{U_{t_q, t_q}^{(0)}(u_q)} U_{\tilde{t}_q, t_q}^{(0)}(u_{q-1}) \right) &= \exp(-i(t_q - \tilde{t}_q)(E(u_q) - E(u_{q-1}))) \\
&\times \left(1 + g(u_q, u_{q-1}; t_q, \tilde{t}_q; \lambda) \right), \tag{6.61}
\end{aligned}$$

for $g(u_q, u_{q-1}; t_q, \tilde{t}_q; \lambda) = e^{i\lambda \int_{\tilde{t}_q}^{t_q} ds(\kappa_s(u_{q-1}) - \kappa_s(u_q))} - 1$ satisfying

$$\|g(\bullet, u_{q-1}; t_q, \tilde{t}_q; \lambda)\|_{H^{3/2+\sigma}} < C_0 \lambda |t_q - \tilde{t}_q| \quad (6.62)$$

as a function of u_q , with C_0 dependent on $\sigma > 0$, but independent of $u_{q-1}, t_q, \tilde{t}_q, \lambda$. The regularity of g is inherited from the pair interaction potential, $\hat{v} \in H^{3/2+\sigma}(\mathbb{T}^3)$ (see (2.18)), and proven as in (5.30).

Similarly as in the proof of Lemma 6.1, a stationary phase argument yields

$$\begin{aligned} & \left| \int du_q \exp(-i(t_q - \tilde{t}_q)(E(u_q) - E(u_{q-1}))) g(u_q, u_{q-1}; t_{q-1}, t_q; \lambda) \right| \\ & \quad < C \lambda \langle t_q - \tilde{t}_q \rangle^{-1/2}. \end{aligned} \quad (6.63)$$

For the integral in \tilde{t}_q , we thus find

$$\begin{aligned} & \int_0^t d\tilde{t}_q \left| \int du_q \left(\overline{U_{\tilde{t}_q, t_q}^{(0)}(u_q)} U_{\tilde{t}_q, t_q}^{(0)}(u_{q-1}) - e^{-i(t_q - \tilde{t}_q)(E(u_q) - E(u_{q-1}))} \right) \right| \\ & \quad < C \lambda t^{1/2}. \end{aligned} \quad (6.64)$$

It is then straightforward to arrive at (6.45). \square

We may now prove Proposition 6.3.

Proof. To establish Proposition 6.3, it suffices to verify that

$$\lim_{\eta \rightarrow 0} \left| \sum_{q=0}^{M(\eta)} \mathcal{U}_{T/\eta^2}^{(basic;q)}(J; f, g) - \sum_{q=0}^{M(\eta)} \mathcal{U}_{T/\eta^2}^{(basic-main;q)}(J; f, g) \right| = 0. \quad (6.65)$$

For the left hand side, we obtain the bound

$$\begin{aligned} & \sum_{q=0}^{M(\eta)} \left| \mathcal{U}_{T/\eta^2}^{(basic;q)}(J; f, g) - \mathcal{U}_{T/\eta^2}^{(basic-main;q)}(J; f, g) \right| \\ & \leq \sum_{q=1}^{M(\eta)} \left[\lambda t^{1/2} (C \eta^2 t \log \frac{1}{\eta})^{q-1} + \eta (C \eta^2 t \log \frac{1}{\eta})^{q-1} \right], \end{aligned} \quad (6.66)$$

using the estimates on the error terms in (6.46). For the choice of parameters

$$\lambda = O(\eta^2), \quad t = O(\eta^{-2}), \quad M(\eta) = \frac{\log \frac{1}{\eta}}{c_0 \log \log \frac{1}{\eta}} \quad (6.67)$$

where we assume that $c_0 > 2$, this is bounded by

$$\begin{aligned} (6.66) & < \eta (C_0 \log \frac{1}{\eta})^{M(\eta)} M(\eta) \\ & \leq \eta \cdot \eta^{-(\frac{\log C_0}{\log \log \frac{1}{\eta}} + \frac{1}{c_0})} \log \frac{1}{\eta} \\ & \leq \eta^{\frac{1}{10}} \end{aligned} \quad (6.68)$$

if η is sufficiently small. This implies (6.65). \square

6.5. Remark on bounds related to nested diagrams. The analysis given above enables us to control the Feynman amplitudes associated to nested diagrams, as was noted in Section 5.1. To this end, we recall that for the Feynman amplitudes belonging to all diagrams containing crossings or nestings are evaluated by first integrating over all time variables s_j , and subsequently integrating over momentum variables u_j . We note that our argument below exhibits the same ordering of integration steps.

Following the definition of nested diagrams in Section 4.2, a nesting subgraph of length m is a progression of m consecutive immediate recollisions connected via a propagator line to two outermost vertices which are mutually contracted. The contribution to the Feynman amplitude associated to this segment of the graph is proportional to

$$\eta^2 \int_{\mathbb{T}^3} du \int_0^\tau ds U_{t_a, t_a+s}^{(m)}(u) \quad (6.69)$$

for some $\tau \leq 1 = T/\eta^2$, where the integration variable u appears nowhere else in the full expression of the Feynman amplitude (for the entire graph). The factor η^2 accounts for the contraction of the two outermost vertices.

Using Lemma 6.2, one straightforwardly verifies that

$$\begin{aligned} & \eta^2 \int_{\mathbb{T}^3} du \left| \int_0^\tau ds U_{t_a, t_a+s}^{(m)}(u) \right| \\ & \leq \eta^2 \int_{\mathbb{T}^3} du \left| \int_0^\tau ds U_{t_a, t_a+s}^{(\text{main}; m)}(u) \right| + \eta^2 \tau \sup_{s, u} |\Delta U_{t_a, t_a+s}^{(m)}(u)| \\ & \leq C^m \eta^2 \int_{\mathbb{T}^3} du \left| \int_0^\tau ds \frac{(\eta^2 s)^m}{m!} U_{t_a, t_a+s}^{(0)}(u) \right| + C^m \eta, \end{aligned} \quad (6.70)$$

where we have recalled that in Lemma 6.2, the values of the parameters are given by $\lambda, \nu = O(\eta^2)$.

We claim that

$$\left| \int_0^\tau ds \frac{s^m}{m!} U_{t_a, t_a+s}^{(0)}(u) \right| < C^m \frac{1}{|E(u)| + \tau^{-1}} \frac{\tau^m}{m!}. \quad (6.71)$$

The proof of (6.71) is similar to the one of Lemma 5.1. If $|E(u)| \leq \tau^{-1}$, the bound is trivially fulfilled. If $|E(u)| > \tau^{-1}$, we define

$$\zeta := \frac{\pi}{|E(u)|}, \quad (6.72)$$

and the intervals $I_j := [j\zeta, (j+1)\zeta]$ with $j \in J := \mathbb{N} \cap [0, \frac{\tau}{\zeta}]$. Then, the left hand side of (6.71) can be bounded by

$$\begin{aligned} & \int_0^\tau ds \frac{s^m}{m!} U_{t_a, t_a+s}^{(0)}(u) \\ &= \sum_{j \in 2\mathbb{N}_0 \cap J} \int_{I_j} ds \left(U_{t_a, t_a+s}^{(0)}(u) \frac{(s+\zeta)^m}{m!} + U_{t_a, t_a+s}^{(0)}(u) \frac{s^m}{m!} \right) \\ &= \sum_{j \in 2\mathbb{N}_0 \cap J} \int_{[0, \zeta]} ds e^{-isE(u)} \left(e^{is\lambda\bar{\kappa}_{t_a+j\zeta, t_a+(j+1)\zeta+s}(u)} \frac{s^m}{m!} \right. \\ &\quad \left. - e^{i(s+\zeta)\lambda\bar{\kappa}_{t_a+(j+1)\zeta, t_a+(j+1)\zeta+s}(u)} \frac{(s+\zeta)^m}{m!} \right). \end{aligned} \quad (6.73)$$

Similarly as in the proof of Lemma 5.1,

$$|e^{is\lambda\bar{\kappa}_{t_a+j\zeta, t_a+j\zeta+s}(u)} \frac{s^m}{m!} - e^{i(s+\zeta)\lambda\bar{\kappa}_{t_a+(j+1)\zeta, t_a+(j+1)\zeta+s}(u)}| < C \lambda \zeta, \quad (6.74)$$

and evidently,

$$0 \leq \frac{(s+\zeta)^m}{m!} - \frac{s^m}{m!} \leq \zeta \frac{(s+\zeta)^{m-1}}{(m-1)!}. \quad (6.75)$$

Accordingly,

$$\begin{aligned} & \left| \int_0^\tau ds U_{t_a, t_a+s}^{(0)}(u) \frac{s^m}{m!} \right| \\ & \leq C \left[\lambda \zeta \int_0^\tau ds \frac{(s+\zeta)^m}{m!} + \zeta \int_0^\tau ds \frac{(s+\zeta)^{m-1}}{(m-1)!} \right] \\ & \leq C \zeta \frac{(2\tau)^m}{m!} \end{aligned} \quad (6.76)$$

for $\lambda, \tau^{-1} \leq O(\eta^2)$. This proves (6.71).

Therefore, we conclude that

$$\begin{aligned} (6.70) & \leq C^m \eta^2 \int du \frac{1}{|E(u)| + \eta^2} + C^m \eta \\ & \leq C^m \eta^2 \log \frac{1}{\eta} + C^m \eta \\ & \leq C^m \eta, \end{aligned} \quad (6.77)$$

The gain of a factor η is crucial, and immediately implies the bounds on nesting subgraphs used in Section 4.2. For a more detailed discussion of nested diagrams in the context of the weakly disordered Anderson model, we refer to [17, 8].

6.6. Decorated ladders and Boltzmann limit. Following the list of steps explained at the beginning of Section 6, we now carry out step 3. Combining Lemma 6.2 and Proposition 6.3, we derive the Boltzmann limit for the sum of decorated (renormalized) ladders, and complete the proof of Theorem 3.1.

For notational convenience, we introduce the multiindices

$$\underline{m}^{(q)} := (m_0, \dots, m_q) , \quad \tilde{\underline{m}}^{(q)} := (\tilde{m}_0, \dots, \tilde{m}_q) \quad (6.78)$$

for fixed $q \in \mathbb{N}$, and

$$|\underline{m}^{(q)}| := m_0 + \dots + m_q , \quad |\tilde{\underline{m}}^{(q)}| = \tilde{m}_0 + \dots + \tilde{m}_q . \quad (6.79)$$

We use $N(\epsilon) = O\left(\frac{\log \frac{1}{\epsilon}}{\log \log \frac{1}{\epsilon}}\right)$ with $\frac{1}{\epsilon} = t = \frac{T}{\eta^2}$ as in (5.39), and consider

$$\begin{aligned} & \mathcal{U}_t^{(ren)}(J; f, g) \\ &:= \sum_{\bar{n}=0}^{N(\epsilon)} \sum_{n+\tilde{n}=2\bar{n}} \sum_{\pi \subset \Gamma_{n,\tilde{n}}^{(ladd)}} \lim_{L \rightarrow \infty} \text{Amp}_{\pi}(J; f, g; t; \eta) \\ &= \sum_{\substack{M_0, \dots, M_{N(\epsilon)} \in \mathbb{N}_0 \\ \sum M_j \leq N(\epsilon)}} \sum_{q \in \mathbb{N}_0} \sum_{\substack{\underline{m}^{(q)}, \tilde{\underline{m}}^{(q)} \\ |\underline{m}^{(q)}| + |\tilde{\underline{m}}^{(q)}| + q = M_q}} \mathcal{U}_t^{(ren;q;\underline{m}^{(q)}, \tilde{\underline{m}}^{(q)})}(J; f, g) \end{aligned} \quad (6.80)$$

where

$$\begin{aligned} & \mathcal{U}_t^{(ren;q;\underline{m}^{(q)}, \tilde{\underline{m}}^{(q)})}(J; f, g) := (-\eta^2)^q \int du_0 \cdots du_q J(u_q) \overline{f(u_0)} g(u_0) \\ & \quad \int_0^t dt_q \cdots \int_0^{t_2} dt_1 \int_0^t d\tilde{t}_q \cdots \int_0^{\tilde{t}_2} d\tilde{t}_1 \prod_{j=0}^q U_{t_{j-1}, t_j}^{(m_j)}(u_j) \overline{U_{\tilde{t}_{j-1}, \tilde{t}_j}^{(\tilde{m}_j)}(u_j)}, \end{aligned} \quad (6.81)$$

using the convention that $t_{-1} = 0 = \tilde{t}_{-1}$. This is the Feynman amplitude of a ladder with q rungs, where the two particle edges labeled by the momentum u_j are decorated with m_j , respectively \tilde{m}_j , immediate recollisions.

To extract the dominant terms in this expression, we define

$$\begin{aligned}
& \mathcal{U}_t^{(ren-main;q;\underline{m}^{(q)},\tilde{\underline{m}}^{(q)})}(J;f,g) \\
:= & (-\eta^2)^q \int_0^t dt_q \cdots \int_0^{t_2} dt_1 \int_0^t d\tilde{t}_q \cdots \int_0^{\tilde{t}_2} d\tilde{t}_1 \int du_0 \cdots du_q J(u_q) \overline{f(u_0)} g(u_0) \\
& \prod_{j=1}^q U_{t_{j-1},t_j}^{(main;m_j)}(u_j) \overline{U_{\tilde{t}_{j-1},\tilde{t}_j}^{(main;\tilde{m}_j)}(u_j)} \\
= & (-\eta^2)^q \int_0^t dt_q \cdots \int_0^{t_2} dt_1 \int_0^t d\tilde{t}_q \cdots \int_0^{\tilde{t}_2} d\tilde{t}_1 \int du_0 \cdots du_q J(u_q) \overline{f(u_0)} g(u_0) \\
& \prod_{j=1}^q U_{t_{j-1},t_j}^{(0)}(u_j;u_0) \overline{U_{\tilde{t}_{j-1},\tilde{t}_j}^{(0)}(u_j;u_0)} \\
& \prod_{j=1}^q \left\{ \frac{1}{m_j!} \left(-\eta^2 (t_j - t_{j-1}) \int du' \frac{1}{E(u') - E(u_j) - i\nu} \right)^{m_j} \right. \quad (6.82) \\
& \left. \frac{1}{\tilde{m}_j!} \left(-\eta^2 (\tilde{t}_j - \tilde{t}_{j-1}) \int du' \frac{1}{E(u') - E(u_j) + i\nu} \right)^{\tilde{m}_j} \right\}
\end{aligned}$$

where

$$U_{t_{j-1},t_j}^{(0)}(u_j;u_0) := U_{t_{j-1},t_j}^{(0)}(u_j) \overline{U_{t_{j-1},t_j}^{(0)}(u_0)}, \quad (6.83)$$

and where we have inserted a factor

$$1 = \overline{U_{0,t}^{(0)}(u_0)} U_{0,t}^{(0)}(u_0) = \prod_{j=1}^q \overline{U_{t_{j-1},t_j}^{(0)}(u_0)} U_{\tilde{t}_{j-1},\tilde{t}_j}^{(0)}(u_0). \quad (6.84)$$

To get (6.82) from (6.81), we have replaced the contributions from the immediate recollisions by their dominant parts identified in Lemma 6.2.

Moreover, we define

$$\begin{aligned}
& \mathcal{U}_t^{(ren-main-0;q;\underline{m}^{(q)},\tilde{\underline{m}}^{(q)})}(J;f,g) \\
:= & (-\eta^2)^q \int_0^t dt_q \cdots \int_0^{t_2} dt_1 \int_0^t d\tilde{t}_q \cdots \int_0^{\tilde{t}_2} d\tilde{t}_1 \int du_0 \cdots du_q J(u_q) \overline{f(u_0)} g(u_0) \\
& \prod_{j=1}^q U_{t_{j-1},t_j}^{(0)}(u_j;u_0) \overline{U_{\tilde{t}_{j-1},\tilde{t}_j}^{(0)}(u_j;u_0)} \\
& \prod_{j=1}^q \left\{ \frac{1}{m_j!} \left(-\eta^2 (t_j - t_{j-1}) \int du' \frac{1}{E(u') - E(u_0) - i\nu} \right)^{m_j} \right. \quad (6.85) \\
& \left. \frac{1}{\tilde{m}_j!} \left(-\eta^2 (\tilde{t}_j - \tilde{t}_{j-1}) \int du' \frac{1}{E(u') - E(u_0) + i\nu} \right)^{\tilde{m}_j} \right\}.
\end{aligned}$$

To get this expression from (6.82), the momenta u_j have been replaced by u_0 in the last product in (6.82). Here, we anticipate the conservation of kinetic energy in the collision processes, which will emerge in the kinetic scaling limit. We then prove the following result which immediately implies Theorem 3.1.

Proposition 6.5. *Let $N(\epsilon)$ be as in Section 5.1.3. Then, for any fixed, finite $T > 0$ and $t = \frac{T}{\eta^2}$, and $\nu = \epsilon = \frac{1}{t}$,*

$$\begin{aligned} \lim_{\eta \rightarrow 0} \sum_{\substack{M_0, \dots, M_{N(\epsilon)} \in \mathbb{N}_0 \\ \sum M_j \leq N(\epsilon)}} \sum_{q \in \mathbb{N}_0} \sum_{\substack{\underline{m}(q), \tilde{\underline{m}}(q) \\ |\underline{m}(q)| + |\tilde{\underline{m}}(q)| + q = M_q}} \mathcal{U}_{T/\eta^2}^{(ren-main-0; q; \underline{m}^{(q)}, \tilde{\underline{m}}^{(q)})}(J; f, g) \\ = \int du \overline{f(u)} g(u) F_T(u) \quad (6.86) \end{aligned}$$

where $F_T(u)$ satisfies the linear Boltzmann equation (3.4) with initial condition $F_0(u) = J(u)$. Moreover,

$$\begin{aligned} \lim_{\eta \rightarrow 0} \sum_{\substack{M_0, \dots, M_{N(\epsilon)} \in \mathbb{N}_0 \\ \sum M_j \leq N(\epsilon)}} \sum_{q \in \mathbb{N}_0} \sum_{\substack{\underline{m}(q), \tilde{\underline{m}}(q) \\ |\underline{m}(q)| + |\tilde{\underline{m}}(q)| + q = M_q}} \left[\mathcal{U}_{T/\eta^2}^{(ren;q; \underline{m}^{(q)}, \tilde{\underline{m}}^{(q)})}(J; f, g) \right. \\ \left. - \mathcal{U}_{T/\eta^2}^{(ren-main; q; \underline{m}^{(q)}, \tilde{\underline{m}}^{(q)})}(J; f, g) \right] = 0, \quad (6.87) \end{aligned}$$

and

$$\begin{aligned} \lim_{\eta \rightarrow 0} \sum_{\substack{M_0, \dots, M_{N(\epsilon)} \in \mathbb{N}_0 \\ \sum M_j \leq N(\epsilon)}} \sum_{q \in \mathbb{N}_0} \sum_{\substack{\underline{m}(q), \tilde{\underline{m}}(q) \\ |\underline{m}(q)| + |\tilde{\underline{m}}(q)| + q = M_q}} \left[\mathcal{U}_{T/\eta^2}^{(ren-main; q; \underline{m}^{(q)}, \tilde{\underline{m}}^{(q)})}(J; f, g) \right. \\ \left. - \mathcal{U}_{T/\eta^2}^{(ren-main-0; q; \underline{m}^{(q)}, \tilde{\underline{m}}^{(q)})}(J; f, g) \right] = 0. \quad (6.88) \end{aligned}$$

Proof. We first verify the Boltzmann limit for the main term before proving the error estimates.

- 1. Proof of (6.86). We have

$$\begin{aligned} \lim_{\eta \rightarrow 0} \sum_{\substack{M_0, \dots, M_{N(\epsilon)} \in \mathbb{N}_0 \\ \sum M_j \leq N(\epsilon)}} \sum_{q \in \mathbb{N}_0} \sum_{\substack{\underline{m}(q), \tilde{\underline{m}}(q) \\ |\underline{m}(q)| + |\tilde{\underline{m}}(q)| + q = M_q}} \mathcal{U}_{T/\eta^2}^{(ren-main-0; q; \underline{m}^{(q)}, \tilde{\underline{m}}^{(q)})}(J; f, g) \\ = \sum_{M_0, \dots, M_j, \dots \in \mathbb{N}_0} \sum_{q \in \mathbb{N}_0} \sum_{\substack{\underline{m}(q), \tilde{\underline{m}}(q) \\ |\underline{m}(q)| + |\tilde{\underline{m}}(q)| + q = M_q}} \mathcal{U}_{T/\eta^2}^{(ren-main-0; q; \underline{m}^{(q)}, \tilde{\underline{m}}^{(q)})}(J; f, g) \\ = \sum_{q \in \mathbb{N}_0} \sum_{\substack{\underline{m}(q), \tilde{\underline{m}}(q) \in \mathbb{N}_0^{q+1}} \mathcal{U}_{T/\eta^2}^{(ren-main-0; q; \underline{m}^{(q)}, \tilde{\underline{m}}^{(q)})}(J; f, g). \quad (6.89) \end{aligned}$$

We find

$$\begin{aligned}
& \sum_{\underline{m}^{(q)}, \tilde{\underline{m}}^{(q)} \in \mathbb{N}_0^{q+1}} \mathcal{U}_t^{(ren-main-0; q; \underline{m}^{(q)}, \tilde{\underline{m}}^{(q)})}(J; f, g) \\
&= (-\eta^2)^q \int_0^t dt_1 \cdots \int_0^{t_2} dt_1 \int_0^t d\tilde{t}_q \cdots \int_0^{\tilde{t}_2} d\tilde{t}_1 \int du_0 \cdots du_q J(u_q) \overline{f(u_0)} g(u_0) \\
&\quad \prod_{j=1}^q U_{t_{j-1}, t_j}^{(0)}(u_j; u_0) \overline{U_{\tilde{t}_{j-1}, \tilde{t}_j}^{(0)}(u_j; u_0)} \\
&\quad \prod_{j=1}^q e^{-\eta^2(t_j - t_{j-1}) \int du' \frac{1}{E(u') - E(u_0) - i\nu}} e^{-\eta^2(\tilde{t}_j - \tilde{t}_{j-1}) \int du' \frac{1}{E(u') - E(u_0) + i\nu}}. \tag{6.90}
\end{aligned}$$

First of all, we note that the expression obtained from setting the product on the last line equal to 1 is precisely $\mathcal{U}_t^{(basic; q)}(J; f, g)$; that is, the amplitude of a basic ladder with q rungs from Section 6.4. Clearly, the product on the last line equals

$$e^{-\eta^2 t 2 \operatorname{Im} \int du' \frac{1}{E(u') - E(u_0) - i\nu}}. \tag{6.91}$$

We are choosing $\nu = \epsilon$, and consider the limit $\nu = \epsilon \rightarrow 0$, where

$$\lim_{\nu \rightarrow 0} \operatorname{Im} \int du' \frac{1}{E(u') - E(u) - i\nu} = \pi \int du' \delta(E(u') - E(u)). \tag{6.92}$$

Thus, it follows straightforwardly from this, and from Proposition 6.3 that for $\lambda = O(\eta^2)$, and any fixed, finite $T > 0$,

$$F_T^{(ren-main-0)}(J; f, g) := \lim_{\eta \rightarrow 0} \sum_{q \in \mathbb{N}_0} \mathcal{U}_{T/\eta^2}^{(ren-main-0; q)}(J; f, g) \tag{6.93}$$

exists, and

$$F_T^{(ren-main-0)}(J; f, g) = \int du \overline{f(u)} g(u) F_T(u) \tag{6.94}$$

where

$$F_T(u) = e^{-2\pi T \int du' \delta(E(u) - E(u'))} F_T^{(basic)}(u) \tag{6.95}$$

(see Section 6.4 for the definition of $F_T^{(basic)}(u)$) satisfies the linear Boltzmann equation

$$\partial_T F_T(u) = 2\pi \int du' \delta(E(u) - E(u')) (F_T(u') - F_T(u)), \tag{6.96}$$

with initial condition

$$F_0(u) = \lim_{L \rightarrow \infty} J(u) = \lim_{L \rightarrow \infty} \frac{1}{L^3} \rho_0(a_u^+ a_u). \tag{6.97}$$

This proves (6.86).

- 2. Proof of (6.87). Recalling Lemma 6.2, we consider

$$\mathcal{U}_{T/\eta^2}^{(ren;q;\underline{m}^{(q)},\tilde{\underline{m}}^{(q)})}(J;f,g) - \mathcal{U}_{T/\eta^2}^{(ren-main;q;\underline{m}^{(q)},\tilde{\underline{m}}^{(q)})}(J;f,g) \quad (6.98)$$

$$\begin{aligned}
&= (-\eta^2)^q \int du_0 \cdots du_q J(u_q) \overline{f(u_0)} g(u_0) \int_0^t dt_q \cdots \int_0^{t_2} dt_1 \int_0^t d\tilde{t}_q \cdots \int_0^{\tilde{t}_2} d\tilde{t}_1 \\
&\quad \left[\prod_{j=1}^q \left(U_{t_{j-1},t_j}^{(m_j)}(u_j) \overline{U_{t_{j-1},t_j}^{(0)}(u_0)} \right) \left(\overline{U_{\tilde{t}_{j-1},\tilde{t}_j}^{(\tilde{m}_j)}(u_j)} U_{\tilde{t}_{j-1},\tilde{t}_j}^{(0)}(u_0) \right) \right. \\
&\quad \left. - \prod_{j=1}^q \left(U_{t_{j-1},t_j}^{(main;m_j)}(u_j) \overline{U_{t_{j-1},t_j}^{(0)}(u_0)} \right) \left(\overline{U_{\tilde{t}_{j-1},\tilde{t}_j}^{(main;\tilde{m}_j)}(u_j)} U_{\tilde{t}_{j-1},\tilde{t}_j}^{(0)}(u_0) \right) \right] \\
&= (A) + (B) \quad (6.99)
\end{aligned}$$

where

$$\begin{aligned}
(A) &:= (-\eta^2)^q \sum_{\ell=0}^q \int du_0 \cdots du_q J(u_q) \overline{f(u_0)} g(u_0) \\
&\quad \int_0^t dt_q \cdots \int_0^{t_2} dt_1 \int_0^t d\tilde{t}_q \cdots \int_0^{\tilde{t}_2} d\tilde{t}_1 \\
&\quad \left[\left(\prod_{j=1}^{\ell-1} U_{t_{j-1},t_j}^{(m_j)}(u_j) \right) \Delta U_{t_{\ell-1},t_\ell}^{(m_j)}(u_\ell) \left(\prod_{j=\ell+1}^q U_{t_{j-1},t_j}^{(main;m_j)}(u_j) \right) \right. \\
&\quad \left. \left[\prod_{j=1}^q \overline{U_{t_{j-1},t_j}^{(0)}(u_0)} \right] \prod_{j=1}^q \left(\overline{U_{\tilde{t}_{j-1},\tilde{t}_j}^{(\tilde{m}_j)}(u_j)} U_{\tilde{t}_{j-1},\tilde{t}_j}^{(0)}(u_0) \right) \right]. \quad (6.100)
\end{aligned}$$

The functions $U_{t_a,t_b}^{(main;m)}(u)$ and $\Delta U_{t_a,t_b}^{(m)}(u)$ were defined in connection with Lemma 6.2. (B) is the analogous term with the roles of the variables t_i and \tilde{t}_i exchanged, and with $U_{t_{j-1},t_j}^{(m)}(u_j)$ replaced by $U_{t_{j-1},t_j}^{(main;m)}(u_j)$.

To bound the integrals with respect to \tilde{t}_j in (6.100), we recall from (6.10) that

$$\begin{aligned}
U_{t_a,t_b}^{(m)}(u) &= (-\eta^2)^m \int_{\mathbb{R}_+^{2m+1}} ds_1 \cdots ds_{2m+1} \delta(t_b - t_a - \sum s_j) \\
&\quad \int_{(\mathbb{T}^3)^{2m}} du_2 \cdots du_{2m+1} \prod_{j=1}^m \delta(u_{2j+1} - u_{2j-1}) \\
&\quad \prod_{\ell=1}^{2m+1} e^{-i \int_{t_{\ell-1}}^{t_\ell} ds'(E(u_\ell) - \lambda \kappa_{s'}(u_\ell))}. \quad (6.101)
\end{aligned}$$

Thus,

$$\begin{aligned}
& \int du_0 \cdots du_q \int_0^t d\tilde{t}_q \cdots \int_0^{\tilde{t}_2} d\tilde{t}_1 \prod_{j=1}^q \left(\overline{U_{\tilde{t}_{j-1}, \tilde{t}_j}^{(m_j)}(u_j)} U_{\tilde{t}_{j-1}, \tilde{t}_j}^{(0)}(u_0) \right) \\
&= \int du_0 \cdots du_q \prod_{j=1}^q \int_0^{\tilde{t}_j - \tilde{t}_{j-1}} d\tilde{s}_j \\
&\quad \prod_{j=1}^q (-\eta^2)^{m_j} \int_{(\mathbb{T}^3)^{2m_j}} du_{j,2} \cdots du_{j,2m_j+1} \prod_{i=1}^{m_j} \delta(u_{j,2i+1} - u_{j,2i-1}) \\
&\quad \int_{\mathbb{R}_+^{2m_j+1}} ds_{j,1} \cdots ds_{j,2m_j+1} \delta(\tilde{s}_j - \sum s_{j,i}) \\
&\quad \prod_{\ell=1}^{2m_j+1} e^{-i \int_{t_{\ell-1}}^{t_\ell} ds' ((E(u_{j,\ell}) - E(u_0)) - \lambda(\kappa_{s'}(u_{j,\ell}) - \kappa_{s'}(u_0)))}. \quad (6.102)
\end{aligned}$$

We now bound this expression by performing the integrals over all time integrations first, using Lemma 5.1, where we begin with the latest time, and successively integrate out the preceding time variable. We obtain that this is bounded by

$$\begin{aligned}
& |(6.102)| \\
&\leq \int du_0 \cdots du_q \prod_{j=1}^q \int_{(\mathbb{T}^3)^{2m_j}} du_{j,2} \cdots du_{j,2m_j+1} \prod_{i=1}^{m_j} \delta(u_{j,2i+1} - u_{j,2i-1}) \\
&\quad \prod_{j=1}^q (-\eta^2)^{m_j} \frac{1}{|E(u_j) - E(u_0)| + \nu} \prod_{\ell=1}^{2m_j+1} \frac{1}{|E(u_{j,\ell}) - E(u_0)| + \nu}. \quad (6.103)
\end{aligned}$$

Integrating out all delta distributions, we obtain

$$\begin{aligned}
& (6.103) \\
&\leq \int du_0 \cdots du_q \left[\prod_{j=1}^q \eta^{2m_j} \left(\frac{1}{|E(u_j) - E(u_0)| + \nu} \right)^{m_j+1} \right. \\
&\quad \left. \left(\int du \frac{1}{|E(u) - E(u_0)| + \nu} \right)^{m_j} \right] \\
&\leq \prod_{j=1}^q (C\eta^2\nu^{-1}(\log \frac{1}{\nu})^2)^{m_j}, \quad (6.104)
\end{aligned}$$

for $\nu = \epsilon = O(\eta^2)$.

From $|U_{\tilde{t}_{j-1}, \tilde{t}_j}^{(0)}(u)| = 1$, and (6.15), we find

$$|U_{\tilde{t}_{j-1}, \tilde{t}_j}^{(main; m_j)}(u_j)| \leq \frac{(C\eta^2(t_j - t_{j-1}))^{m_j}}{m_j!} < e^{C\eta^2(t_j - t_{j-1})}, \quad (6.105)$$

so that

$$\prod_{j=\ell+1}^q |U_{\tilde{t}_{j-1}, \tilde{t}_j}^{(main; m_j)}(u_j)| \leq e^{C\eta^2 \sum_j (t_j - t_{j-1})} < e^{C\eta^2 t} < C'. \quad (6.106)$$

since $T = \eta^2 t$ is fixed, and hence of order $O(1)$ in η .

Moreover, Lemma 6.2 implies that

$$\begin{aligned} |U_{t_{j-1}, t_j}^{(m_j)}(u_j)| &\leq \frac{(C\eta^2(t_j - t_{j-1}))^{m_j}}{m_j!} + \eta m_j (c')^{m_j} \\ &< e^{C\eta^2(t_j - t_{j-1})} + \eta c^{m_j}, \end{aligned} \quad (6.107)$$

(where we may, for instance, assume $c = 2c' > 1$) so that

$$\begin{aligned} \prod_{j=1}^{\ell-1} |U_{t_{j-1}, t_j}^{(m_j)}(u_j)| &< \prod_{j=1}^q (e^{C\eta^2(t_j - t_{j-1})} + \eta c^{m_j}) \\ &< e^{C\eta^2 \sum_j (t_j - t_{j-1})} \prod_{j=1}^q (1 + \eta c^{m_j}) \\ &< e^{C\eta^2 \sum_j (t_j - t_{j-1})} (1 + \sum_{j=1}^q \binom{q}{r} \eta^r c^{\sum_j m_j}) \\ &< e^{C\eta^2 t} (1 + 2\eta c^{M_q}) < C'. \end{aligned} \quad (6.108)$$

For the second inequality, we used $e^{C\eta^2(t_j - t_{j-1})} \geq 1$ and $c \geq 1$. For the last inequality, we have recalled that $M_q \leq N(\epsilon) = \frac{C|\log \eta|}{\log |\log \eta|}$, from the beginning of Section 6.6. This implies that

$$\eta c^{M_q} \leq \eta \eta^{-c \frac{1}{\log |\log \eta|}} < \eta^{\frac{1}{10}} \quad (6.109)$$

for η sufficiently small. Moreover, $T = \eta^2 t$ is fixed and of order $O(1)$ in η . This implies that (6.108) is bounded uniformly in η , for η sufficiently small.

Hence, we conclude that

$$\begin{aligned} |(A)| &< (C\eta^2)^q \sum_{\ell=0}^q \int_0^t dt_q \cdots \int_0^{t_2} dt_1 \|J\|_\infty \|f\|_\infty \|g\|_\infty \\ &\quad |\Delta U_{t_{\ell-1}, t_\ell}^{(m_\ell)}(u_\ell)| \prod_{j=1}^q (C\eta^2 \nu^{-1} (\log \frac{1}{\nu})^2)^{\tilde{m}_j}. \end{aligned} \quad (6.110)$$

Using the bounds (6.16),

$$\begin{aligned} |(A)| &< (C\eta^2)^q \|J\|_\infty \|f\|_\infty \|g\|_\infty \sum_{\ell=0}^q \int_0^t dt_q \cdots \int_0^{t_2} dt_1 \\ &\quad \left(m_\ell \lambda \nu^{-1/2} (\eta^2 \nu^{-1})^{m_\ell} + \nu^{1/2} (\eta^2 \nu^{-1})^{m_\ell} \right) \\ &\quad \prod_{j=1}^q (C\eta^2 \nu^{-1} (\log \frac{1}{\nu})^2)^{\tilde{m}_j}, \end{aligned} \quad (6.111)$$

$$\begin{aligned} &< (C \log \frac{1}{\eta})^q \|J\|_\infty \|f\|_\infty \|g\|_\infty \frac{(\eta^2 t)^q}{q!} \\ &\quad \sum_{\ell=0}^q \left(m_\ell \lambda \nu^{-1/2} (\eta^2 \nu^{-1})^{m_\ell} + \nu^{1/2} (\eta^2 \nu^{-1})^{m_\ell} \right) \\ &\quad \prod_{j=1}^q (C\eta^2 \nu^{-1} (\log \frac{1}{\nu})^2)^{m_j}. \end{aligned} \quad (6.112)$$

We notice the crucial gain of the factors $\lambda\nu^{-1/2}$ and $\nu^{1/2}$.

The term (B) can be estimated in a similar way.

In conclusion, we arrive at

$$\begin{aligned}
& \left| \sum_{\substack{M_0, \dots, M_{N(\epsilon)} \in \mathbb{N}_0 \\ \sum M_j \leq N(\epsilon)/2}} \sum_{q \in \mathbb{N}_0} \sum_{\substack{\underline{m}^{(q)}, \tilde{\underline{m}}^{(q)}; |\underline{m}^{(q)}| + |\tilde{\underline{m}}^{(q)}| + q = M_q}} \left[(A) + (B) \right] \right| \\
& < \sum_{\substack{M_0, \dots, M_{N(\epsilon)} \in \mathbb{N}_0 \\ \sum M_j \leq N(\epsilon)/2}} \sum_{q \leq N(\epsilon)} M_q (\lambda \nu^{-1/2} + \nu^{1/2}) (C \eta^2 \nu^{-1} (\log \frac{1}{\nu})^2)^{M_q} \\
& < (2N(\epsilon) + 2)! (\lambda \nu^{-1/2} + \nu^{1/2}) (C \eta^2 \nu^{-1} (\log \frac{1}{\nu})^2)^{N(\epsilon)} \\
& < \eta^{1/10}
\end{aligned} \tag{6.113}$$

for the choice of parameters of Section 5.1.3, and for $\nu = \epsilon$. This proves (6.87).

- 3. Proof of (6.88). To begin with, we note that

$$\begin{aligned}
& \int du' \frac{1}{E(u') - E(u_j) - i\nu} - \int du' \frac{1}{E(u') - E(u_0) - i\nu} \\
& = (E(u_j) - E(u_0)) m_j G_1(u_0, u_j; \nu)
\end{aligned} \tag{6.114}$$

where

$$\begin{aligned}
G_1(u_0, u_j; \nu) &:= \int du' \frac{1}{E(u') - E(u_0) - i\nu} \frac{1}{E(u') - E(u_j) - i\nu} \\
&= \int_{\mathbb{R}_+ \times \mathbb{R}_+} ds_1 ds_2 \int du' e^{-is_1(E(u') - E(u_0) - i\nu)} e^{-is_2(E(u') - E(u_j) - i\nu)}.
\end{aligned} \tag{6.115}$$

From a stationary phase argument,

$$\begin{aligned}
|G_1(u_0, u_j; \nu)| &\leq \int_{\mathbb{R}_+ \times \mathbb{R}_+} ds_1 ds_2 \langle s_1 + s_2 \rangle^{-3/2} e^{-\nu(s_1 + s_2)} \\
&= \int_{s \geq s' \geq 0} ds' ds \langle s \rangle^{-3/2} e^{-\nu s} \\
&\leq \int_{s \geq 0} ds \langle s \rangle^{-1/2} e^{-\nu s} \\
&\leq C \nu^{-1/2}.
\end{aligned} \tag{6.116}$$

More generally, for $m_j \in \mathbb{N}$, one can straightforwardly show along the same lines that

$$\begin{aligned}
& \prod_{j=0}^q \left(\int du' \frac{1}{E(u') - E(u_j) - i\nu} \right)^{m_j} - \prod_{j=0}^q \left(\int du' \frac{1}{E(u') - E(u_0) - i\nu} \right)^{m_j} \\
& = \sum_{\ell=0}^q (E(u_\ell) - E(u_0)) G_{\ell; \underline{m}^{(q)}}(\underline{u}^{(q)}; \nu)
\end{aligned} \tag{6.117}$$

with $\underline{u}^{(q)} := (u_0, \dots, u_q)$, for functions

$$\begin{aligned} G_{\ell; \underline{m}^{(q)}}(\underline{u}^{(q)}; \nu) &:= \left(\prod_{j=0}^{\ell-1} (R_{i\nu}(u_j))^{m_j} \right) \left(\prod_{j=\ell+1}^q (R_{i\nu}(u_0))^{m_j} \right) \\ &\quad \cdot G_1(u_0, u_\ell; \nu) \sum_{i'=0}^{m_\ell-1} (R_{i\nu}(u_\ell))^{m_\ell-i'} (R_{i\nu}(u_0))^{i'} \end{aligned} \quad (6.118)$$

with $R_{i\nu}(u) := \int du' \frac{1}{E(u') - E(u) - i\nu}$. One easily sees that, for $\ell \in \{0, \dots, q\}$,

$$|G_{\ell; \underline{m}^{(q)}}(\underline{u}^{(q)}; \nu)| < m_\ell C^{\sum_{j=0}^q m_j} \nu^{-1/2}, \quad (6.119)$$

for a constant C independent of ℓ , ν and $\{m_j\}$, using (6.27) and (6.116).

Next, we observe that

$$\begin{aligned} &(E(u_j) - E(u_0)) U_{t_{j-1}, t_j}^{(0)}(u_j; u_0) \\ &= (i \partial_{s_j} e^{-is_j(E(u_j) - E(u_0))}) e^{i\lambda \int_{t_{j-1}}^{t_j} ds' (\kappa_{s'}(u_j) - \kappa_{s'}(u_0))} \end{aligned} \quad (6.120)$$

where $s_j = t_j - t_{j-1} \geq 0$.

Therefore, one finds

$$\begin{aligned} &\mathcal{U}_t^{(ren-main; q; \underline{m}^{(q)}, \tilde{\underline{m}}^{(q)})}(J; f, g) - \mathcal{U}_t^{(ren-main-0; q; \underline{m}^{(q)}, \tilde{\underline{m}}^{(q)})}(J; f, g) \\ &= (I) + (II) + (III) + (IV) \end{aligned} \quad (6.121)$$

with

$$\begin{aligned} (I) &:= i(-\eta^2)^q \sum_{\ell=0}^q \int_0^t dt_q \cdots \int_0^{t_{\ell+2}} dt_{\ell+1} \int_0^{t_{\ell+1}} dt_{\ell-1} \cdots \int_0^{t_2} dt_1 \\ &\quad \int_0^t d\tilde{t}_q \cdots \int_0^{\tilde{t}_2} d\tilde{t}_1 \int du_0 \cdots du_q J(u_q) \overline{f(u_0)} g(u_0) G_{\ell; \underline{m}^{(q)}}(\underline{u}^{(q)}; \nu) \\ &\quad \left[\prod_{j=1}^q U_{t_{j-1}, t_j}^{(0)}(u_j; u_0) \overline{U_{\tilde{t}_{j-1}, \tilde{t}_j}^{(0)}(u_j; u_0)} \right. \\ &\quad \left. \frac{1}{m_j!} \left(-\eta^2 (t_j - t_{j-1}) \right)^{m_j} \right. \\ &\quad \left. \frac{1}{\tilde{m}_j!} \left(-\eta^2 (\tilde{t}_j - \tilde{t}_{j-1}) \int du' \frac{1}{E(u') - E(u_j) + i\nu} \right)^{\tilde{m}_j} \right] \Big|_{s_\ell=0}^{t_{\ell+1}-t_{\ell-1}} \end{aligned} \quad (6.122)$$

and

$$\begin{aligned}
(II) := & -i(-\eta^2)^q \sum_{j=0}^q \int_0^t dt_q \cdots \int_0^{t_j} ds_j \int_0^{t_{j-1}=t_j-s_j} dt_{j-2} \int_0^{t_2} dt_1 \\
& \int_0^t d\tilde{t}_q \cdots \int_0^{\tilde{t}_2} d\tilde{t}_1 \int du_0 \cdots du_q J(u_q) \overline{f(u_0)} g(u_0) G_{j;\underline{m}^{(q)}}(\underline{u}^{(q)}; \nu) \\
& \left\{ \prod_{\ell=1}^q \overline{U_{\tilde{t}_{\ell-1}, \tilde{t}_\ell}^{(0)}(u_\ell; u_0)} \right. \\
& \frac{1}{\tilde{m}_\ell!} \left(-\eta^2 (\tilde{t}_\ell - \tilde{t}_{\ell-1}) \int du' \frac{1}{E(u') - E(u_\ell) + i\nu} \right)^{\tilde{m}_\ell} \left. \right\} \\
& \left\{ \prod_{i=1}^{j-1} U_{t_{i-1}, t_i}^{(0)}(u_i; u_0) \frac{1}{m_i!} (-\eta^2 (t_i - t_{i-1}))^{m_i} \right\} \\
& \partial_{s_j} \left[\frac{(\eta^2 s_j)^{m_j}}{m_j!} e^{i\lambda \int_{t_{j-1}}^{t_j} ds' (\kappa_{s'}(u_j) - \kappa_{s'}(u_0))} \right. \\
& \left. \left\{ \prod_{i=j+1}^q U_{t_{i-1}, t_i}^{(0)}(u_i; u_0) \frac{1}{m_i!} (-\eta^2 (t_i - t_{i-1}))^{m_i} \right\} \right]. \tag{6.123}
\end{aligned}$$

where for each ℓ , integration by parts has been applied to the variable s_ℓ . With respect to the latter, the expression (6.122) comprises the boundary terms.

The terms (III) and (IV) are similar to (I) and (II), but in (III) and (IV), integration by parts is applied to the variables $\tilde{s}_j = \tilde{t}_j - \tilde{t}_{j-1}$. Accordingly, the roles of s_j and t_i are exchanged with those of \tilde{s}_j and \tilde{t}_i , respectively, and moreover, $\prod(\int du' \frac{1}{E(u') - E(u_j) - i\nu})^{m_j}$ is exchanged with $\prod(\int du' \frac{1}{E(u') - E(u_0) - i\nu})^{m_j}$.

Bounds on term (I). Clearly,

$$|(I)| \leq \eta^{2q} \|J\|_\infty \|f\|_\infty \|g\|_\infty \sum_{\ell=0}^q \sup_{\underline{u}^{(q)}} |G_{\ell;\underline{m}^{(q)}}(\underline{u}^{(q)}; \nu)| A_1 A_2 \tag{6.124}$$

where

$$\begin{aligned}
A_1 := & \int du_0 \cdots du_q \left| \int_0^t d\tilde{t}_q \cdots \int_0^{\tilde{t}_2} d\tilde{t}_1 \prod_{j=1}^q \left[\overline{U_{\tilde{t}_{j-1}, \tilde{t}_j}^{(0)}(u_j; u_0)} \right. \right. \\
& \left. \left. \frac{1}{\tilde{m}_j!} \left(\eta^2 (\tilde{t}_j - \tilde{t}_{j-1}) \int du' \frac{1}{E(u') - E(u_j) + i\nu} \right)^{\tilde{m}_j} \right] \right| \tag{6.125}
\end{aligned}$$

and

$$\begin{aligned}
A_2 := & \sup_{u_0, \dots, u_q} \int_0^t dt_1 \cdots \int_0^{t_{\ell+2}} dt_{\ell+1} \int_0^{t_\ell} dt_{\ell-1} \cdots \int_0^{t_2} dt_1 \\
& \prod_{j=1}^q \left\{ \frac{1}{m_j!} \left(\eta^2 (t_j - t_{j-1}) \right)^{m_j} \right\}_{s_\ell=0}^{t_\ell}. \tag{6.126}
\end{aligned}$$

We use (6.71) and bounds similar as in the case of (6.102), to estimate the factor (6.125) involving the integrals in \tilde{t}_j and u_j . Thereby, we obtain an upper bound

$$A_1 < (C \log \frac{1}{\nu})^{\sum \tilde{m}_j}. \quad (6.127)$$

Furthermore, we note that for every fixed index ℓ in the sum, there are only $q - 1$ integrals with respect to the variables t_i , keeping in mind that there is no integration over t_ℓ . Accordingly, we bound the integrals in t_j in A_2 by $\frac{t^{q-1}}{(q-1)!}$ (the gain of a factor $\frac{1}{t}$ as compared to t^q is crucial). Moreover, it is evident that

$$\sum_{m_0, \dots, m_q} \prod_{j=1}^q \frac{1}{m_j!} \left(\eta^2 (t_j - t_{j-1}) \right)^{m_j} = e^{\eta^2 \sum (t_j - t_{j-1})} = e^{\eta^2 t} < C \quad (6.128)$$

holds for the integrand in A_2 . Recalling the bound on $|G_{\ell; \underline{m}^{(q)}}(\underline{u}^{(q)}; \nu)|$ in (6.119), we straightforwardly obtain

$$\begin{aligned} & \sum_{1 \leq |\underline{m}^{(q)}| + |\tilde{\underline{m}}^{(q)}| < M_q} |(I)| \\ & < \sum_{1 \leq |\underline{m}^{(q)}| + |\tilde{\underline{m}}^{(q)}| < M_q} q m_\ell \eta^{2q} \nu^{-1/2} \frac{t^{q-1}}{(q-1)!} C^{\sum m_j} (C' \log \frac{1}{\nu})^{1+\sum \tilde{m}_j} \\ & < q \frac{M^{2q}}{(2q-1)!} \eta^2 \nu^{-1/2} \frac{(C \eta^2 t)^{q-1}}{(q-1)!} (C' \log \frac{1}{\nu})^{M_q} \end{aligned} \quad (6.129)$$

where the factor q accounts for the sum with respect to $\ell = 0, \dots, q$, for each fixed $\underline{m}^{(q)}$ and $\tilde{\underline{m}}^{(q)}$. Moreover, we have used the estimate

$$\#\{(\underline{m}^{(q)}, \tilde{\underline{m}}^{(q)}) \mid |\underline{m}^{(q)}| + |\tilde{\underline{m}}^{(q)}| < M_q\} \leq C \sum_{r \leq M_q} \frac{r^{2q-1}}{(2q-1)!} < \frac{C M_q^{2q}}{(2q-1)!} \quad (6.130)$$

from $\#\{(\underline{m}^{(q)}, \tilde{\underline{m}}^{(q)}) \mid |\underline{m}^{(q)}| + |\tilde{\underline{m}}^{(q)}| = r\} \leq C \frac{r^{2q-1}}{(2q-1)!}$; the latter bounds the number of lattice points in a simplex in \mathbb{N}_0^{2q} of side length r .

Hence, we conclude that, for any fixed $T = \eta^2 t > 0$ and η sufficiently small,

$$\begin{aligned} \sum_{1 \leq |\underline{m}^{(q)}| + |\tilde{\underline{m}}^{(q)}| < M_q} |(I)| & < q \eta \frac{(\eta^2 t)^{q-1}}{(q-1)!} (C (\log \frac{1}{\epsilon})^3)^{N(\epsilon)} \\ & < q \eta \epsilon^{-3/10} \frac{T^{q-1}}{(q-1)!} \\ & < \eta^{1/2} \end{aligned} \quad (6.131)$$

using that $M_q < N(\epsilon) = \frac{\log \frac{1}{\epsilon}}{10 \log \log \frac{1}{\epsilon}}$ (see Section 5.1.3), $\nu = \epsilon = O(\eta^2)$, and $q \geq 1$.

Bounds on term (II). For the term $\partial_{s_j} [\dots]$ in (6.123), We note that $\partial_{s_j} f(t_\ell) = f'(t_\ell)$ for every $\ell \geq j$, since $t_\ell = s_0 + \dots + s_j + \dots + s_\ell$. For $i > j$,

$$\begin{aligned} & \partial_{s_j} U_{t_{i-1}, t_i}^{(0)}(u_i; u_0) \\ &= \partial_{s_j} e^{-i \int_{t_{i-1}}^{t_i} ds' (E(u_i) - E(u_0) - \lambda(\kappa_{s'}(u_i) - \kappa_{s'}(u_0)))} \\ &= i \lambda (\kappa_{s'}(u_i) - \kappa_{s'}(u_0)) \Big|_{s'=t_{i-1}}^{t_i} U_{t_{i-1}, t_i}^{(0)}(u_i; u_0). \end{aligned} \quad (6.132)$$

There is no term proportional to $(E(u_i) - E(u_0))$ on the last line because $\partial_{s_j} t_i = 1 = \partial_{s_j} t_{i-1}$. Similarly,

$$\partial_{s_j} (t_i - t_{i-1})^{m_i} = 0 \quad (6.133)$$

whenever $i > j$. Using the a priori bound $\|\kappa_s\|_{L^\infty(\mathbb{T}^3)} < c$, uniformly in s , (a consequence of the fermion statistics, as we recall), we conclude that

$$\begin{aligned} \left| \partial_{s_j} [\dots] \text{ in (6.123)} \right| &< C \prod_{i>j} \frac{(\eta^2(t_i - t_{i-1}))^{m_i}}{m_i!} \\ &\quad \left(\eta^2 \frac{(\eta^2(t_j - t_{j-1}))^{m_{j-1}}}{(m_j - 1)!} + \lambda \frac{(\eta^2(t_j - t_{j-1}))^{m_j}}{(m_j)!} \right), \end{aligned} \quad (6.134)$$

where $0 \leq t_i - t_{i-1} \leq t = O(\eta^{-2})$. Combined with (6.102), we arrive at

$$\sum_{1 \leq |\underline{m}^{(q)}| + |\tilde{\underline{m}}^{(q)}| < M_q} |(II)| < \frac{M_q^{2q}}{(2q-1)!} (\eta^2 + \lambda) (c \eta^2 t)^q (c \log \frac{1}{\nu})^{M_q}, \quad (6.135)$$

where the gain of a factor $(\eta^2 + \lambda)$ is crucial. Using the same arguments as above for (6.131), we find

$$\sum_{1 \leq |\underline{m}^{(q)}| + |\tilde{\underline{m}}^{(q)}| < M_q} |(II)| < \eta^{1/2} \quad (6.136)$$

for every fixed $T = \eta^2 t > 0$, for η sufficiently small, given that $M_q < N(\epsilon) = \frac{\log \frac{1}{\epsilon}}{10 \log \log \frac{1}{\epsilon}}$, $\nu = \epsilon = O(\eta^2)$, and $q \geq 1$.

Concluding the proof. The terms (III) and (IV) are estimated similarly, and yield similar bounds as those derived for (I) and (II), respectively. In conclusion, we find that for any $T = \eta^2 t$ fixed and η sufficiently small,

$$\begin{aligned} & \left| \sum_{\substack{M_0, \dots, M_{N(\epsilon)} \in \mathbb{N}_0 \\ \sum M_j \leq N(\epsilon)}} \sum_{q \in \mathbb{N}_0} \sum_{\substack{\underline{m}^{(q)}, \tilde{\underline{m}}^{(q)} \\ |\underline{m}^{(q)}| + |\tilde{\underline{m}}^{(q)}| + q = M_q}} \left[(I) + (II) + (III) + (IV) \right] \right| \\ &< (N(\epsilon))^{N(\epsilon)} 4 \eta^{1/2} \sum_{q \in \mathbb{N}_0} \chi(q < N(\epsilon)) \\ &< 2 \eta^{1/2} (N(\epsilon))^{N(\epsilon)+2} \\ &< \eta^{1/3}, \end{aligned} \quad (6.137)$$

for the choice of parameters given in Section 5.1.3; that is, with $\lambda, \nu = O(\eta^2)$, and $N(\epsilon) = \frac{\log \frac{1}{\epsilon}}{10 \log \log \frac{1}{\epsilon}}$. Here, we have used the trivial but sufficient bound

$$\#\{(M_0, \dots, M_{N(\epsilon)}) \in \mathbb{N}_0^{N(\epsilon)} \mid \sum M_j \leq N(\epsilon)\} < (N(\epsilon))^{N(\epsilon)}, \quad (6.138)$$

and observed that in the sum with respect to q , the conditions $\sum M_j \leq N(\epsilon)$ and $|\underline{m}^{(q)}| + |\tilde{\underline{m}}^{(q)}| + q = M_q$ imply that $q \leq N(\epsilon)$. In conclusion, (6.88) follows. \square

This completes our proof of Theorem 3.1.

7. PROOF OF THEOREM 3.2

Based on our proof of Theorem 3.1, the proof of Theorem 3.2 is straightforward. Applying Lemma 5.1 with $\zeta = \lambda^{1+\delta} = \eta^2$, and $\delta > 0$ arbitrary but fixed, we immediately conclude that the estimates for crossing and nesting diagrams used in the proof of Theorem 3.1 remain valid.

Adapting the proof of Theorem 3.1 to the relative scaling of parameters as asserted in Theorem 3.2, yields the bound

$$\lim_{L \rightarrow \infty} |\text{Amp}_\pi(f, g; \epsilon; \eta)| \leq C(J, f, g) (\log \frac{1}{\lambda})^4 (c\eta^2 \lambda^{-1} \bar{n} \log \frac{1}{\epsilon})^{\bar{n}} \quad (7.1)$$

(see (5.31) for comparison). Since $\lambda = \eta^{2-\delta}$, we find

$$\begin{aligned} \sum_{1 \leq \bar{n} \leq N} \sum_{\pi \in \Gamma_{2\bar{n}}} \lim_{L \rightarrow \infty} |\text{Amp}_\pi(f, g; \epsilon; \eta)| \\ < N! (\log \frac{1}{\lambda})^4 (c\lambda^\delta N \log \frac{1}{\lambda})^N. \end{aligned} \quad (7.2)$$

In contrast to (5.34), we are not carrying out any classification of Feynman graphs. The sum (7.2) extends over all classes of graphs, including ladder, crossing and nesting graphs.

For the given scaling of parameters, we note that instead of the overall factor $\epsilon^{1/5} \approx \lambda^{(1+\delta)/10}$ in (5.34), we are now obtaining a factor $\eta^{\delta N}$ which is $\ll \lambda^{(1+\delta)/10}$ for λ sufficiently small, and for parameters chosen similarly as in Section 5.1.3.

In particular, we can now substitute $\lambda^{\frac{1+\delta}{2}}$ for η , in the bounds given in Section 5.1.3, so that $N = O(\frac{\log \frac{1}{\lambda}}{\log \log \frac{1}{\lambda}})$.

Likewise, the estimates on the Duhamel remainder term of Section 5.1.2 can be easily adapted to the present case, and we find that

$$\begin{aligned} \int dp \overline{f(p)} g(p) F_T(p) &= \lim_{\lambda \rightarrow 0} \lim_{L \rightarrow \infty} \mathbb{E} [\rho_{T/\lambda}(a^+(f) a(g))] \\ &= \lim_{L \rightarrow \infty} \rho_0(a^+(f) a(g)) \end{aligned} \quad (7.3)$$

for any $T > 0$. That is, for any initial state ρ_0 satisfying the assumptions of the theorem, the Boltzmann limit F_T is stationary. This proves Theorem 3.2. \square

8. PROOF OF THEOREM 3.4

Theorem 3.4 also follows as an almost immediate consequence of our proof of Theorem 3.1. Since we are assuming that $F \in L^\infty(\mathbb{T}^3)$ does not depend on the time variable, we have

$$U_{s_1, s_2}(u) = \exp \left(i((s_2 - s_1)(E(u) - \lambda(\widehat{v} * F)(u))) \right), \quad (8.1)$$

which yields a time-independent shift of the kinetic energy,

$$E(p) \rightarrow \tilde{E}_\lambda(p) := E(p) - \lambda(\hat{v} * F)(p). \quad (8.2)$$

Thus, under the assumptions that (3.17) and the crossing estimate (3.18) hold, the derivation of the Boltzmann limit reduces to the case treated in [10] for $\lambda = 0$, with $\tilde{E}_\lambda(p)$ replacing the kinetic energy function $E(p)$.

In the kinetic scaling limit determined by $t = \frac{T}{\eta^2}$ and $\eta \rightarrow 0$, one accordingly obtains

$$\lim_{\eta \rightarrow 0} \mathcal{G}[F; \eta; \lambda; T/\eta^2; f, g] = \int dp \overline{f(p)} g(p) \tilde{F}_T(p) \quad (8.3)$$

for the given choice of F , where \tilde{F}_T satisfies the equation

$$\partial_T \tilde{F}_T(p) = 2\pi \int du \delta(\tilde{E}_\lambda(u) - \tilde{E}_\lambda(p)) (\tilde{F}_T(u) - \tilde{F}_T(p)) \quad (8.4)$$

with initial condition $\tilde{F}_0(p) = F(p)$. While (8.4) has the form of a linear Boltzmann equation, \tilde{F}_T is an auxiliar quantity of which only the stationary solutions are relevant for our discussion. Here, we point out that the renormalized energy \tilde{E}_λ in the collision kernel in (8.4) is determined by F , not by \tilde{F}_T ; see (8.2).

Stationary solutions of (8.4) are determined by the condition $\partial_T \tilde{F}_T(p) = 0$, so that $\tilde{F}_T(p) = \tilde{F}_0(p) = F(p)$. This holds if and only if $F(p)$ satisfies the self-consistency condition

$$F(p) = \frac{1}{\tilde{m}_\lambda(p)} \int du \delta(\tilde{E}_\lambda(u) - \tilde{E}_\lambda(p)) F(u), \quad (8.5)$$

where

$$\tilde{m}_\lambda(p) := 2\pi \int du \delta(\tilde{E}_\lambda(u) - \tilde{E}_\lambda(p)). \quad (8.6)$$

This proves Theorem 3.4. \square

Acknowledgements. T.C. thanks I. Sasaki for discussions about related subjects, and J.L. Lebowitz for inspiring comments. The work of T.C. was supported by NSF grant DMS-0704031 / DMS-0940145.

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